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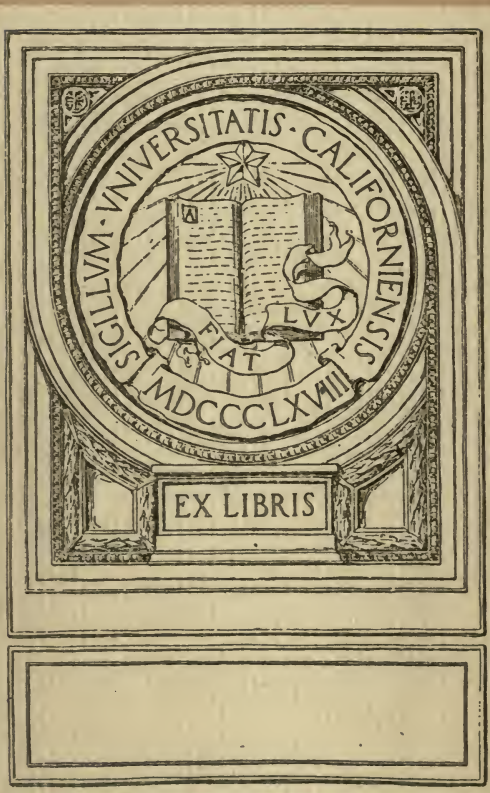
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LESSONS INTRODUCTORY TO THE MODERN HIGHER ALGEBRA.

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LESSONS

INTRODUCTORY TO THE

MODERN HIGHER ALGEBRA.

BY

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TO

A. CAYLEY, ESQ., AND J. J. SYLVESTER, ESQ.,

I BEG TO INSCRIBE

THIS ATTEMPT TO RENDER SOME OF THEIR DISCOVERIES BETTER KNOWN,

In Acknowledgment

OF

THE OBLIGATIONS I AM UNDER, NOT ONLY TO THEIR PUBLISHED WRITINGS,

BUT ALSO

TO THEIR INSTRUCTIVE CORRESPONDENCE.

165039

PREFACE.

THE pressure of other engagements having prevented me from taking any part in the preparation of this new edition of my *Higher Algebra*, I have to express my obligations to the good offices of my friend Mr. Cathcart in revising the work and superintending its progress through the press. A comparison of the number of pages will show that he has made several additions to the Contents of the last edition. These will chiefly be found in "Applications to binary quantics," which are now divided into Lessons XVII., XVIII.

GEO. SALMON.

TRINITY COLLEGE, DUBLIN,
May, 1885.

CONTENTS.

LESSON I.

DETERMINANTS.—PRELIMINARY ILLUSTRATIONS AND DEFINITIONS.

PAGE

Rule of signs 5
Sylvester's umbral notation 8

LESSON II.

REDUCTION AND CALCULATION OF DETERMINANTS.

Minors [called by Jacobi partial determinants] . . . 10
Examples of reduction 13
Product of differences of n quantities expressed as a determinant . 14
Reduction of bordered Hessians 17
Continuants 18

LESSON III.

MULTIPLICATION OF DETERMINANTS.

The theorem stated as one of linear transformation . . . 21
Extension of the theorem 22
Examples of multiplication of determinants . . . 23
Product of squares of differences of n quantities . . . 23
Radius of sphere circumscribing a tetrahedron . . 26
Relation connecting mutual distances of points on a circle or sphere . 26
Of five points in space 27
Sylvester's proof that equation of secular inequalities has all roots real . 28

LESSON IV.

MINOR AND RECIPROCAL DETERMINANTS.

Relations connecting products of determinants . . . 29
Solution of a system of linear equations 29
Reciprocal systems 30
Minors of reciprocal system expressed in terms of those of the original . 30
Minors of a determinant which vanishes 32
Forms for expanding a determinant of the fourth order . . 32

LESSON V.

SYMMETRICAL AND SKEW SYMMETRICAL DETERMINANTS.

Differentials of a determinant with respect to its constituents . . 34
If a symmetric determinant vanishes, the same bordered is a perfect square 37
Skew symmetric determinants of odd degree vanish . . . 37
Of even degree are perfect squares 38
Nature of the square root [see Jacobi, *Crelle*, II. 354; XXIX. 236] 39
Orthogonal substitutions 41
Number of terms in a symmetrical determinant . . . 45

LESSON VI.

DISCRIMINATING SYMMETRICAL DETERMINANTS.

PAGE
New proof that equation of secular inequalities has all its roots real . 48
Sylvester's expressions for Sturm's functions in terms of the roots . . 49
Borchardt's proof 54

LESSON VII.

SYMMETRIC FUNCTIONS.

Newton's formulæ for sums of powers of roots . . . 57
Improvement of this process 57
Determinant expression for sums of powers . . . 57
Rules for weight and order of a symmetric function . . . 58
Formula for sum of powers of differences of roots . . . 60
Differential equation of functions of differences . . . 61
Symmetric functions of homogeneous equations . . . 62
Differential equation when binomial coefficients are used . . 64
Serret's notation 65

LESSON VIII.

ORDER AND WEIGHT OF ELIMINANTS

Eliminants defined 66
Elimination by symmetric functions 67
Order and weight of resultant of two equations . . . 69
Symmetric functions of common values for a system of two equations . 72
Extension of principles to any number of equations . . . 75

LESSON IX.

EXPRESSION OF ELIMINANTS AS DETERMINANTS.

Elimination by process for greatest common measure . . . 76
Euler's method 77
Conditions that two equations should have two common factors . . 78
Sylvester's dialytic method 79
Bezout's method 81
Cayley's statement of it 83
Jacobians defined 84
Jacobian and derived equations satisfied by common system which satisfies equations of same degree 84
Expression by determinants, in particular cases, of resultant of three equations 85
Cayley's method of expressing resultants as quotients of determinants . 87

LESSON X.

DETERMINATION OF COMMON ROOTS.

Expression of roots common to a system of equations by the differentials of the resultant 91
Equations connecting these differentials when the resultant vanishes . 91
Expressions by the minors of Bezout's matrix . . . 93
General expression for differentials of resultant with respect to any quantities entering into the equations 96
General conditions that a system may have two common roots . 97

LESSON XI.

DISCRIMINANTS.

	PAGE
Order and weight of discriminants	99
Discriminant expressed in terms of the roots	101
Discriminant of product of two or more functions	101
Discriminant is of form $a_0\phi + a_1{}^2\psi$	102
Formation of discriminants by the differential equation	103
Method of finding the equal roots when the discriminant vanishes	104
Extension to any number of variables	106
Discriminant of a quadratic function	107

LESSON XII.

LINEAR TRANSFORMATIONS.

Invariance of discriminants	108
Number of independent invariants	110
Invariants of systems of quantics	112
Covariants	114
Every invariant of a covariant is an invariant of the original	114
Invariants of emanants are covariants	116
Contravariants	117, 120
Differential symbols are contragredient to variables	119
$x\xi + y\eta$ + &c. absolutely unaltered by transformation	120
Mixed concomitants	121
Evectants	122
Evectant of discriminant of a quantic whose discriminant vanishes	123

LESSON XIII.

FORMATION OF INVARIANTS AND COVARIANTS.

Method by symmetric functions	124
Concomitants which vanish when two or more roots are equal	125
Method of mutual differentiation of covariants and contravariants	126
Differential coefficients substituted for the variables in a contravariant give covariants	127
For binary quantics, covariants and contravariants not essentially distinct	127
Invariants and covariants of second order in coefficients	129
Cubinvariant of a quartic	129
Every quantic of odd degree has an invariant of the 4^{th} order	129
Cubicovariant of cubic	130
Method of the differential equation	130
Weight of an invariant of given order	130
Binary quantics of odd degree cannot have invariants of odd order	130
Coefficients of covariants determined by the differential equation	131
Skew invariants	131
Investigation of number of independent invariants by the differential equation	132
Source of product of two covariants is product of their sources	134
Cayley's definition of covariants	135
Extension to any number of variables	136

LESSON XIV.

SYMBOLICAL REPRESENTATION OF CONCOMITANTS.

PAGE
Method of formation by derivative symbols . . . 137
Order of derivative in coefficients and in the variables . . . 140
Table of invariants of the third order 141
Hermite's law of reciprocity 142
Derivative symbols for ternary quantics 144
Symbols for evectants 146
Method of Aronhold and Clebsch 147

LESSON XV.

CANONICAL FORMS.

Generality of a form examined by its number of constants . . 150
Reduction of a quadratic function to a sum of squares . . 151
Principle that the number of negative squares is unaffected by real substitution 151
Reduction of cubic to its canonical form 152
Discriminant of a cubic and of its Hessian differ only in sign . 153
General reduction of quantic of odd degree 153
Methods of forming canonizant 154, 155
Condition that a quantic of order $2n$ be reducible to a sum of n, $2n^{th}$ powers . 156
Canonical forms for quantics of even order . . . 157
Canonical forms for sextic and octavic 159
For ternary and quaternary cubics . . . 160

LESSON XVI.

SYSTEMS OF QUANTICS.

Combinants defined, differential equation satisfied by them . . 161
Number of double points in an involution 162
Geometrical interpretation of Jacobian 162
Factor common to two quantics is square factor in Jacobian . . 162
Order of condition that $u + kv$ may have cubic factor . . . 163
Nature of discriminant of Jacobian 164
Discriminant of discriminant of $u + kv$ 165
Proof that resultant is a combinant 166
Discriminant with respect to x, y, of a function of u, v . . . 167
Discriminant of discriminant of $u + kv$ for ternary quantics . . 169
Tact-invariant of two curves 169
Tact-invariant of complex curves 170
Osculants 171
Covariants of a binary system connected with those of a ternary . 172

LESSON XVII.

APPLICATIONS TO BINARY QUANTICS.

Invariants when said to be distinct 175
Number of independent covariants 175
Cayley's method of forming a complete system . . . 177
THE QUADRIC 178
Resultant of two quadrics 180

PAGE
SYSTEM OF THREE OR MORE QUADRICS 181
Extension to quantics in general of theorems concerning quadrics . 182
THE CUBIC 183
Geometric meaning of covariant cubic 184
Square of this cubic expressed in terms of the other covariants . 186, 192
Solution of cubic 186
SYSTEM OF CUBIC AND QUADRIC 187, 225
Geometrical illustrations 189
THE QUARTIC 189
Catalecticants 190
Discriminant of a quartic 190
Relation of covariants of cubic derived from that of invariants of a quartic . 191
Sextic covariant geometrical meaning of 193
Relations connecting quadratic factors of 194
Reduction of quartic to its canonical form . . . 194
Relation connecting covariants of quartic 195
Symmetrical solution of quartic 196
Criteria for real and imaginary roots 197
The quartic can be brought to its canonical form by real substitutions . 197
Conditions that a quartic should have two square factors . . 198
Cayley's proof that the system of invariants and covariants is complete . 199
Application of Burnside's method 200
Covariants of system of quartic and its Hessian . . . 201
Hessian of Hessian of any quantic 202
SYSTEM OF QUADRIC AND QUARTIC 202, 267
SYSTEM OF TWO CUBICS 204
Resultant of the system 205
Condition that $u + \lambda v$ may be a perfect cube 205
Mode of dealing with equations which contain a superfluous variable . 207
Jacobian and simplest linear covariants 209
Any two cubics may be regarded as differential coefficients of same quartic . 210
Invariants of invariants of $u + \lambda v$ are combinants . . . 211
Process of obtaining concomitant of system from concomitant of single quantic 212
Complete list of covariants of system 213
Plane geometrical illustration of system of two cubics . . . 214
SYSTEM OF FOUR CUBICS 215
Illustration of twisted cubics 216
SYSTEM OF QUARTIC AND CUBIC 218, 226
SYSTEM OF TWO QUARTICS 219
Their resultant 220
Condition that $u + \lambda v$ should be perfect square . . . 220
Condition that $u + \lambda v$ should have cubic factor . . . 221
Special form when both quartics are sums of two fourth powers . . 223
Three quadrics derived functions of a single quartic . . . 224
Three quadrics quadric covariants of two cubics . . . 225

LESSON XVIII.

APPLICATIONS TO HIGHER BINARY QUANTICS.

THE QUINTIC 227
Canonical form of quintic 227
Condition that two quartics be first differentials of the same quintic . 228

PAGE
Discriminant of quintic 229
Fundamental invariants of quintic 232
Conditions for two pairs of equal roots 233
All invariants of a quantic vanish if more than half its roots be all equal . 233
Hermite's canonical form 233
Hermite's skew invariant 233, 282
Its geometrical meaning 234
Covariants of quintic for canonical form 235
Cayley's arrangement of these forms 237
Cayley's canonical form 239
Sign of discriminant of any quantic determines whether it has an odd or even number of pairs of imaginary roots 239
Criteria furnished by Sturm's theorem for a quintic 240
If roots all real, canonizant has imaginary factors 241
Invariant expression of criteria for real roots 241
Sylvester's criteria 245
Conditions involving variation within certain limits of a constant . 246
Cayley's modification of Sylvester's method 248
Hermite's forme-type 249
The Tschirnhausen transformation 250
Modified by Hermite and Cayley 251
Applied to quartic 252
Applied to quintic 254
Sextic resolvent of a quintic 257
Harley's and Cockle's resolvent 257
Expression of invariants in terms of roots 258
THE SEXTIC—its invariants and simplest covariants . . 260
Conditions for cubic factor or for two square factors . . . 263
The discriminant 263
Simplest quartic covariant 266
Quadric covariants 268
The skew invariant expressed in terms of other invariants . . 269
Functions likely to afford criteria for real roots . . . 271
SYSTEM OF TWO QUARTICS 271
Jacobian identified with any sextic by means of a quintic . . 273
Functional determinant of three quartics 274
Can be similarly identified 275
New canonical forms of sextic by Brill 276
Also by Stephanos 277
Factors of discriminant of Jacobian 278
Sextic covariant of third order in coefficients 279
Canonical form referred to ternary system 281
Condition for sextic to be sextic-covariant of quartic . . 282
to be Hessian of quintic 282

LESSON XIX.

ON THE ORDER OF RESTRICTED SYSTEMS OF EQUATIONS.

Order and weight of systems defined 284
Restricted systems 285
Determinant systems, k rows, $k+1$ columns 287
Order and weight of conditions that two equations have two common roots . 291

PAGE
System of conditions that three equations should have a common root . 293
Systems of conditions that equation have cubic factor or double square factor 294
Intersection of quantics having common curves 295
Case of distinct common curves 297
Number of quadrics passing through five points and touching four planes . 298
Rank of curve represented by a system of k rows, $k+1$ columns . 299
System of conditions that two equations should have three common roots . 300
System of quantics having a surface common . . . 301
Having common surface and curve 303
Having common two surfaces 305
System of conditions that three ternary quantics have two common points 306, 309
Rule when the constants in systems of equations are connected by relations 307
Number of curve triplets having two common points . . 310
Mr. S. Roberts' method 310

LESSON XX.

APPLICATIONS OF SYMBOLICAL METHODS.

Symbolical expression for invariant or covariant . . . 314
Clebsch's proof that every covariant can be so expressed . . 315
Formulæ of transformation 316
Reduction to standard forms 318
Transvection 320
Symbolical expression for derivative of derivative . . . 321
Forms of any order obtained by transvection from forms of lower order 323
Gordan and Clebsch's proof that the number of irreducible covariants is finite 324
Every invariant symbol has $(ab)^p$ as a factor, where p is at least half n . 326
Symbolical expression for resultant of quadratic and any equation . 326
Investigation of equation of inflexional tangents to cubic . . 330
Application of symbolical forms to theory of double tangents to plane curves . 333
Typical exposition of an even binary quantic . . . 335
of a quantic of order $3p$ 337

NOTES.

History of determinants 338
Commutants 339
On rational functional determinants 340
Hessians 341
Symmetric functions 342
Elimination 342
Discriminants 342
Bezoutiants 343
Linear transformations 343
Canonical forms 345
Combinants 345
Applications to binary quantics 345
Table of transvectants 346
M. Roberts' table of sums of powers of differences . . . 347
Table of resultants 348
Hirsch and Cayley's tables of symmetric functions . . . 350
Index 357

Errata.

PAGE

40, line 1 after first determinant, *insert* "conjugate" *before* "imaginary."

63, Art. 62 line 4, and p. 64 line 2, write *for* $\lambda, \lambda y$: also in third line following *for* $\lambda, \lambda x$.

135, last sentence of Art. 148, interchange "order in the coefficients" with "degree in the variables."

139, line 4 from end of Art. 153, *read* "coefficient".

196, line 12, add after "thus" "the squares of the factors of U are the values of": also in Ex. 3 and p. 201 Ex. 2 *read* "the squares of the factors."

215, line 5, after H_1 put a semicolon.

226, Ex. 10, *for* Art. "216" *read* "219*b*."

236, line 10 from bottom, *add* "of T *and* U."

243, line 2 begins, "$h\phi'(\alpha) + \varepsilon\psi(\alpha)$."

247, line 2, *read* "$+\frac{1}{5}$ *JD*.", as on p. 260.

269, Add as footnote to Art. 260, on E as given in the second Edition, "where the following corrections should be made
p. 259, col. 3 line 6, coefficient -96 should be $+24$,
p. 265 „ 3 „ 13, „ -360 „ „ $+20300$."

279, Ex. 3, at end *add* "of Art. 266."

279, Ex. 4, the third determinant form for j should have $-$ before it.

280, Ex. 5, line 2, *read* "(Art. 257)."

300, Art. 285, lines 1 and 8, *for* "the last article" *read* "Art. 272."

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LESSONS ON HIGHER ALGEBRA.

LESSON I.

DETERMINANTS.—PRELIMINARY ILLUSTRATIONS.

1. If we are given n homogeneous equations of the first degree between n variables, we can eliminate the variables, and obtain a result involving the coefficients only, which is called the *determinant* of those equations. We shall, in what follows, give rules for the formation of these determinants, and shall state some of their principal properties; but we think that the general theory will be better understood if we first give illustrations of its application to the simplest examples.

Let us commence, then, with two equations between two variables

$$a_1 x + b_1 y = 0, \quad a_2 x + b_2 y = 0.$$

Adding the first equation multiplied by b_2 to the second multiplied by $-b_1$, we get $a_1 b_2 - a_2 b_1 = 0$, the left-hand member of which is the determinant required. The ordinary notation for this determinant is

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}.$$

We shall, however, often, for brevity, write $(a_1 b_2)$ to express this determinant, leaving the reader to supply the term with the negative sign; and in this notation it is obvious that $(a_1 b_2) = +(a_2 b_1)$. The coefficients a_1, b_1, &c., which enter into the expression of a determinant, are called the *constituents* of that determinant, and the products $a_1 b_2$, &c., are called the *elements* of the determinant.

2. It can be verified at once that we should have obtained the same result if we had eliminated the variables between the equations

$$a_1x + a_2y = 0, \quad b_1x + b_2y = 0.$$

In other words

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix},$$

or the value of the determinant is not altered if we write the horizontal rows vertically, and *vice versâ*.

3. If we are given two homogeneous equations between three variables,

$$a_1x + a_2y + a_3z = 0, \quad b_1x + b_2y + b_3z = 0;$$

these equations are sufficient to determine the mutual ratios of x, y, z. Thus, by eliminating y and x alternately, we can express x and y in terms of z, when we find

$$(a_1b_2)\,x = (a_2b_3)\,z, \quad (a_1b_2)\,y = (a_3b_1)\,z.$$

In other words, x, y, z are proportional respectively to (a_2b_3), (a_3b_1), (a_1b_2). Substituting these values in the original equations, we obtain the identical relations

$$a_1(a_2b_3) + a_2(a_3b_1) + a_3(a_1b_2) = 0, \quad b_1(a_2b_3) + b_2(a_3b_1) + b_3(a_1b_2) = 0;$$

relations which are verified at once by writing them at full length, as for instance

$$a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0.$$

The notation $\left\| \begin{matrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{matrix} \right\|$

(where the number of columns is greater than the number of rows) is used to express the three determinants which can be obtained by suppressing in turn each one of the columns, viz. the three determinants of which we have been speaking, (a_2b_3), (a_3b_1), (a_1b_2).

4. Let us now proceed to a system of three equations

$$a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0, \quad a_3x + b_3y + c_3z = 0.$$

Then, if we multiply the first by (a_2b_3), the second by (a_3b_1), the third by (a_1b_2), and add, the coefficients of x and y will vanish

in virtue of the identical relations of Art. 3, and the determinant required is

$$c_1(a_2b_3) + c_2(a_3b_1) + c_3(a_1b_2);$$

or, writing at full length,

$$c_1a_2b_3 - c_1a_3b_2 + c_2a_3b_1 - c_2a_1b_3 + c_3a_1b_2 - c_3a_2b_1.$$

It may also be written in either of the forms

$$a_1(b_2c_3) + a_2(b_3c_1) + a_3(b_1c_2), \quad b_1(c_2a_3) + b_2(c_3a_1) + b_3(c_1a_2).$$

This determinant is expressed by the notation

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix},$$

though we shall often use for it the abbreviation $(a_1b_2c_3)$.

It is useful to observe that

$$(a_2b_3c_1) = (a_1b_2c_3), \text{ but } (a_1b_3c_2) = -(a_1b_2c_3).$$

For, by analogy of notation,

$(a_2b_3c_1) = a_2(b_3c_1) + a_3(b_1c_2) + a_1(b_2c_3)$, which is the same as $(a_1b_2c_3)$, while

$(a_1b_3c_2) = a_1(b_3c_2) + a_3(b_2c_1) + a_2(b_1c_3)$, which is the same as $-(a_1b_2c_3)$.

5. We should have obtained the same result of elimination if we had eliminated between the three equations

$$a_1x + a_2y + a_3z = 0, \quad b_1x + b_2y + b_3z = 0, \quad c_1x + c_2y + c_3z = 0.$$

For if, proceeding on the same system as before, we multiply the first equation by (b_2c_3), the second by (c_2a_3), and the third by (a_2b_3), and add, then the coefficients of y and z vanish, and the determinant is obtained in the form

$$a_1(b_2c_3) + b_1(c_2a_3) + c_1(a_2b_3),$$

which, expanded, is found to be identical with $(a_1b_2c_3)$. Hence

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix},$$

or the determinant is not altered by writing the horizontal rows vertically, and *vice versâ*; a property which will be proved to be true of every determinant.

6. Using the notation

$$\left\| \begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \end{matrix} \right\|$$

to denote the system of determinants obtained by omitting in turn each one of the columns, these four determinants are connected by the relations

$$a_1(a_2b_3c_4) - a_2(a_3b_4c_1) + a_3(a_4b_1c_2) - a_4(a_1b_2c_3) = 0,$$
$$b_1(a_2b_3c_4) - b_2(a_3b_4c_1) + b_3(a_4b_1c_2) - b_4(a_1b_2c_3) = 0,$$
$$c_1(a_2b_3c_4) - c_2(a_3b_4c_1) + c_3(a_4b_1c_2) - c_4(a_1b_2c_3) = 0.$$

These relations may be either verified by actual expansion of the determinants, or else may be proved by a method analogous to that used in Art. 3. Take the three equations

$$a_1x + a_2y + a_3z + a_4w = 0,$$
$$b_1x + b_2y + b_3z + b_4w = 0,$$
$$c_1x + c_2y + c_3z + c_4w = 0.$$

Then (as in Art. 5) we can eliminate y and z by multiplying the equations by (b_2c_3), (c_2a_3), (a_2b_3), respectively, and adding, when we find $(a_1b_2c_3)x + (a_4b_2c_3)w = 0.$

In like manner, multiplying by (b_3c_1), (c_3a_1), (a_3b_1) respectively, we get $(a_2b_3c_1)y + (a_4b_3c_1)w = 0.$

And in like manner, $(a_3b_1c_2)z + (a_4b_1c_2)w = 0.$

Now, attending to the remarks about signs (Art. 4), these equations are equivalent to

$(a_1b_2c_3)x = -(a_2b_3c_4)w$, $(a_1b_2c_3)y = (a_3b_4c_1)w$, $(a_1b_2c_3)z = -(a_4b_1c_2)w$,

or x, y, z, w are respectively proportional to $(a_2b_3c_4)$, $-(a_3b_4c_1)$, $(a_4b_1c_2)$, $-(a_1b_2c_3)$; substituting which values in the original equations we obtain the identities already written.

7. If now we have to eliminate between the four equations

$$a_1x + b_1y + c_1z + d_1w = 0,$$
$$a_2x + b_2y + c_2z + d_2w = 0,$$
$$a_3x + b_3y + c_3z + d_3w = 0,$$
$$a_4x + b_4y + c_4z + d_4w = 0,$$

we have only to multiply the first by $(a_2b_3c_4)$, the second by $-(a_3b_4c_1)$, the third by $(a_4b_1c_2)$, the fourth by $-(a_1b_2c_3)$, and add, when the coefficients of x, y, z vanish identically, and the determinant is found to be

$$d_1(a_2b_3c_4) - d_2(a_3b_4c_1) + d_3(a_4b_1c_2) - d_4(a_1b_2c_3);$$

or, writing it at full length, and altering all signs,

$$\begin{aligned} &a_1b_2c_3d_4 - a_1b_3c_2d_4 + a_2b_3c_1d_4 - a_2b_1c_3d_4 + a_3b_1c_2d_4 - a_3b_2c_1d_4 \\ &+ a_1b_4c_2d_3 - a_1b_2c_4d_3 + a_4b_2c_1d_3 - a_4b_1c_2d_3 + a_2b_1c_4d_3 - a_2b_4c_1d_3 \\ &+ a_3b_4c_1d_2 - a_3b_1c_4d_2 + a_4b_1c_3d_2 - a_4b_3c_1d_2 + a_1b_3c_4d_2 - a_1b_4c_3d_2 \\ &+ a_2b_4c_3d_1 - a_2b_3c_4d_1 + a_4b_3c_2d_1 - a_4b_2c_3d_1 + a_3b_2c_4d_1 - a_3b_4c_2d_1. \end{aligned}$$

8. There is no difficulty in extending to any number of equations the process here employed; and the reader will observe that the general expression for a determinant is $\Sigma \pm a_1b_2c_3d_4$ &c., where each product must include all the varieties of the n letters and of the n suffixes, without repetition or omission, and the determinant contains all the 1.2.3...n such products which can be formed. With regard to the sign to be affixed to each element of the determinant, the following is the rule: We give the sign + to the term $a_1b_2c_3d_4$ &c., obtained by reading the determinant from the left-hand top to the right-hand bottom corner; and then "*the sign* + or − *is affixed to each other product according as it is derived from this leading term by an even or odd number of permutations of suffixes.*" Thus, in the last example, the second term $a_1b_3c_2d_4$ differs from the first only by a permutation of the suffixes of b and c; it therefore has an opposite sign. The third term, $a_2b_3c_1d_4$, differs from the second by a permutation of the suffixes of a and c; it therefore has an opposite sign to the second, but it has the same sign with the first term, since it can be derived from it by *twice* permuting suffixes.

Ex. In the determinant $(a_1b_2c_3d_4e_5)$, what sign is to be prefixed to the element $a_3b_5c_2d_1e_4$?

From the first term, permuting the suffixes of a and c, we get $a_3b_2c_1d_4e_5$, the first constituent of which is the same as that in the given term; next permuting the suffixes of b and e, we get $a_3b_5c_1d_4e_2$, which has two constituents the same as the given term; next, permuting c and e, we get $a_3b_5c_2d_4e_1$; lastly, permuting d and e, we get the given term $a_3b_5c_2d_1e_4$. Since, then, there has been an even number (four) of permutations, the sign of the term is +. In fact, the signs of the series of terms are

$$a_1b_2c_3d_4e_5 - a_2b_2c_1d_4e_5 + a_3b_5c_1d_4e_2 - a_3b_5c_2d_4e_1 + a_3b_5c_2d_1e_4.$$

The rule of signs may otherwise be presented thus: we take for each suffix so often as it comes after a superior suffix the sign $-$, and compound these into a single sign $+$ or $-$. Thus comparing the elements $a_1b_2c_3d_4e_5$, $a_3b_5c_2d_1e_4$, it will be seen that the suffix 1 which came first in the former element, is in the latter preceded by three constituents; that the suffix 2 is preceded by two which came after it before, and the suffix 4 by one. The total number of displacements is therefore six, and this being an even number, the sign of the term is positive. Thus the rule is, that the sign of the term is positive when the total number of displacements, as compared with the order in the leading term, is even, and *vice versâ*. The same results will be obtained if, writing the suffixes always in the order 1, 2, 3, &c., we permute the letters, giving to each arrangement of the letters its proper sign $+$ or $-$ according to the rule of signs. Thus the determinant of Art. 7 might be written $a_1b_2c_3d_4 - a_1c_2b_3d_4 + c_1a_2b_3d_4 -$ &c.

9. *A cyclic interchange of suffixes alters the sign when the number of factors in the product is even, but not so when the number of factors is odd.* Thus a_2b_1, being got from a_1b_2 by one interchange of suffixes, has a different sign; but $a_2b_3c_1$ has the same sign with $a_1b_2c_3$, from which it is derived by a double permutation. For, changing the suffixes of a and b, $a_1b_2c_3$ becomes $a_2b_1c_3$, and changing the suffixes of b and c, this again becomes $a_2b_3c_1$. In like manner $a_2b_3c_4d_1$ has an opposite sign to $a_1b_2c_3d_4$, being derived from it by a triple permutation, viz. through the steps $a_2b_1c_3d_4$, $a_2b_3c_1d_4$, $a_2b_3c_4d_1$.

This rule enables us easily to write down the terms of a determinant with their proper signs, by taking the cyclic permutations of each arrangement. Thus, for three rows the arrangements of suffixes are evidently $+\,123$, $+\,231$, $+\,312$, and $-\,213$, $-\,132$, $-\,321$. For four rows the arrangements are

$$+1234 - 2341 + 3412 - 4123\,; \quad -1243 + 2431 - 4312 + 3124\,;$$

$$-1324 + 3241 - 2413 + 4132\,; \quad +1423 - 4231 + 2314 - 3142.$$

10. We are now in a position to replace our former definition of a determinant by another, which we make the foundation

of the subsequent theory. In fact, since a determinant is only a function of its constituents a_1, b_1, c_1, &c., and does not contain the variables x, y, z, &c., it is obviously preferable to give a definition which does not introduce any mention of equations between these quantities x, y, z.

*Let there be n^2 quantities arrayed in a square of n columns and n rows, then the sum with proper signs (as explained, Art. 8) of all possible products of n constituents, one constituent being taken from each column and each row, is called the determinant of these quantities, and is said to be of the n^{th} order. Constituents are said to be *conjugate* to each other, when the place which either occupies in the horizontal rows is the same as that which the other occupies in the vertical columns. A determinant is said to be *symmetrical* when the conjugate constituents are equal to each other; for example,

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

11. In these first lessons, as in the previous examples, we usually write all the constituents in the same row with the same letter, and those in the same column with the same suffix. A common notation, however, is to write the constituents of a determinant with a double suffix, one suffix denoting the row and the other the column, to which the constituent belongs. Thus the determinant of the third order would be written

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3} \\ a_{2,1}, & a_{2,2}, & a_{2,3} \\ a_{3,1}, & a_{3,2}, & a_{3,3} \end{vmatrix},$$

or else

$$\Sigma \pm a_{1,1} a_{2,2} a_{3,3},$$

* We might have commenced with this definition of a determinant, the preceding articles being unnecessary to the scientific development of the theory. We have thought, however, that the illustrations there given would make the general theory more intelligible; and also that the importance of the study of determinants would more clearly appear, when it had been shown that every elimination of the variables from a system of equations of the first degree, and every solution of such a system, gives rise to determinants, such systems of equations being of constant occurrence in every department of pure and applied mathematics.

where, in the sum, the suffixes are interchanged in all possible ways. The preceding notation is occasionally modified by the omission of the letter a, and the determinant is written

$$\begin{vmatrix} (1,1), & (1,2), & (1,3) \\ (2,1), & (2,2), & (2,3) \\ (3,1), & (3,2), & (3,3) \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 11, & 12, & 13 \\ 21, & 22, & 23 \\ 31, & 32, & 33 \end{vmatrix}.$$

Again, Dr. Sylvester has suggested what he calls an *umbral* notation. Consider, for example, the determinant

$$\begin{vmatrix} a\alpha, & b\alpha, & c\alpha, & d\alpha \\ a\beta, & b\beta, & c\beta, & d\beta \\ a\gamma, & b\gamma, & c\gamma, & d\gamma \\ a\delta, & b\delta, & c\delta, & d\delta \end{vmatrix},$$

the constituents of which are $a\alpha$, $b\alpha$, &c., where a, b, c, &c., are not quantities, but, as it were, shadows of quantities; that is to say, have no meaning separately, and only acquire one in combination with one of the other class of umbræ α, β, γ, &c. Thus, for example, if α, β, γ, δ represent the suffixes 1, 2, 3, 4, the constituents in the notation we have ourselves employed are all formed by combining one of the letters a, b, c, d with one of the figures 1, 2, 3, 4. Now the above determinant is written by Dr. Sylvester more compactly

$$\begin{vmatrix} a, & \alpha \\ b, & \beta \\ c, & \gamma \\ d, & \delta \end{vmatrix},$$

which denotes the sum of all possible products of the form $a\alpha.b\beta.c\gamma.d\delta$, obtained by giving the terms in the second column every possible permutation, and changing sign according to the foregoing rule of signs. Observe that if the two columns are identical, and if in general rs means the same thing as sr, then the determinant is symmetrical.

LESSON II.

REDUCTION AND CALCULATION OF DETERMINANTS.

12. We have in the last Lesson given the rule for the formation of determinants, and exemplified some of their properties in particular cases. We shall in this Lesson prove these properties in general, together with some others, which are most frequently used in the reduction and calculation of determinants.

The value of a determinant is not altered if the vertical columns be written horizontally, and vice versâ (see Arts. 2, 5).

This follows immediately from the law of formation (Art. 10), which is perfectly symmetrical with respect to the columns and rows. One of the principal advantages of the notation with double suffixes is that it exhibits most distinctly the symmetry which exists between the horizontal and vertical lines.

13. *If any two rows (or two columns) be interchanged, the sign of the determinant is altered.*

For the effect of the change is evidently a single permutation of two of the letters (or of two of the suffixes), which by the law of formation causes a change of sign.*

14. *If two rows (or if two columns) be identical, the determinant vanishes.*

For these two rows being interchanged, we ought (Art. 13) to have a change of sign, but the interchange of two identical lines can produce no change in the value of the determinant.

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* It may be remarked that a determinant is a function which is determined (except for a common factor) by the properties that it is linear in respect of the constituents of each row and of each column, and that it merely changes sign if two rows or columns be interchanged. Thus for two rows, the most general lineo-linear function of the rows and columns is

$$b_1 (Aa_1 + Ba_2) + b_2 (Ca_1 + Da_2);$$

and the condition that it is to change sign when we interchange a_1 and b_1, a_2 and b_2, gives $A = D = 0$, $B + C = 0$. The function is therefore $C (a_1 b_2 - a_2 b_1)$, and if we agree that the coefficient of $a_1 b_2$ is to be unity, the function is $a_1 b_2 - a_2 b_1$ as before.

Its value, then, does not alter when its sign is changed; that is to say, it is $= 0$.

This theorem also follows immediately from the definition of a determinant, as the result of elimination between n linear equations. For that elimination is performed by solving for the variables from $n-1$ of the equations, and substituting the values so found in the n^{th}. But if this n^{th} equation be the same as one of the others, it must vanish identically when these values are substituted in it.

15. *If every constituent in any row (or in any column) be multiplied by the same factor, then the determinant is multiplied by that factor.*

This follows at once from the fact that every term in the expansion of the determinant contains, as a factor, one, and but one, constituent belonging to the same row or to the same column.

Thus, for example, since every element of the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

contains either a_1, a_2, or a_3, the determinant can be written in the form $a_1A_1 + a_2A_2 + a_3A_3$ (where neither A_1, A_2, nor A_3 contains any constituent from the a column); and if a_1, a_2, a_3 be each multiplied by the same factor k, the determinant will be multiplied by that factor.

Cor. If the constituents in one row or column differ only by a constant multiplier from those in another row or column, the determinant vanishes. Thus

$$\begin{vmatrix} ka_2, & a_2, & a_3 \\ kb_2, & b_2, & b_3 \\ kc_2, & c_2, & c_3 \end{vmatrix} = k \begin{vmatrix} a_2, & a_2, & a_3 \\ b_2, & b_2, & b_3 \\ c_2, & c_2, & c_3 \end{vmatrix} = 0 \text{ (Art. 14).}$$

16. If in any determinant we erase any number of rows and the same number of columns, the determinant formed with the remaining rows and columns is called a *minor* of the given determinant. The minors formed by erasing one row and one

column may be called first minors; those formed by erasing two rows and two columns, second minors, and so on.

We have, in the last article, observed that if the constituents of one column of a determinant be a_1, a_2, a_3, &c., the determinant may be written in the form $a_1A_1 + a_2A_2 + a_3A_3 +$ &c. And it is evident that A_1 is the minor obtained by erasing the line and column which contain a_1, &c. For every element of the determinant which contains a_1 can contain no other constituent from the column a or the line (1); and a_1 must be multiplied by all possible combinations of products of $n-1$ constituents, taken one from each of the other rows and columns. But the aggregate of these form the minor A_1. Compare Art. 7. In like manner the determinant may be written $a_1A_1 + b_1B_1 + c_1C_1 +$ &c., where B_1 is the minor formed by erasing the row and column which contain b_1.

17. *If all the constituents but one vanish in any row or column of a determinant of the n^{th} order, its calculation is reduced to the calculation of a determinant of the $n-1^{th}$ order.* For, evidently, if a_2, a_3, &c., all vanish, the determinant $a_1A_1 + a_2A_2 +$ &c., reduces to the single term a_1A_1; and A_1 is a determinant having one row and one column less than the given determinant.

Conversely, a determinant of the $n-1^{th}$ order may be written as one of the n^{th} or higher order. Thus

$$\begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0 \\ 0, & b_2, & c_2 \\ 0, & b_3, & c_3 \end{vmatrix} \text{ or } = \begin{vmatrix} 1, & b_1, & c_1 \\ 0, & b_2, & c_2 \\ 0, & b_3, & c_3 \end{vmatrix} \text{ &c.}$$

18. *If every constituent in any row (or in any column) be resolvable into the sum of two others, the determinant is resolvable into the sum of two others.*

This follows from the principle used in Art. 16. Thus, if in the Example there given, we write $a_1 + \alpha_1$ for a_1; $b_1 + \beta_1$ for b_1; $c_1 + \gamma_1$ for c_1; then the determinant becomes

$$(a_1 + \alpha_1) A_1 + (b_1 + \beta_1) B_1 + (c_1 + \gamma_1) C_1$$
$$= \{a_1A_1 + b_1B_1 + c_1C_1\} + \{\alpha_1A_1 + \beta_1B_1 + \gamma_1C_1\}.$$

Thus we have

$$\begin{vmatrix} a_1+\alpha_1, & a_2, & a_3 \\ b_1+\beta_1, & b_2, & b_3 \\ c_1+\gamma_1, & c_2, & c_3 \end{vmatrix} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1, & a_2, & a_3 \\ \beta_1, & b_2, & b_3 \\ \gamma_1, & c_2, & c_3 \end{vmatrix}.$$

In like manner, if the constituents in any one column were each the sum of any number of others, the determinant could be resolved into the same number of others.

19. If again, in the preceding, the constituents in the second column were also each the sum of others (if, for instance, we were to write for a_2, $a_2+\alpha_2$; for b_2, $b_2+\beta_2$; for c_2, $c_2+\gamma_2$), then each of the determinants on the right-hand side of the last equation could be resolved into the sum of others; and we see, without difficulty, that

$$(a_1+\alpha_1,\ b_2+\beta_2,\ c_3) = (a_1b_2c_3)+(a_1\beta_2c_3)+(\alpha_1b_2c_3)+(\alpha_1\beta_2c_3).$$

And if each of the constituents in the first column could be resolved into the sum of m others, and each of those of the second into the sum of n others, then the determinant could be resolved into the sum of mn others. For we should first, as in the last Article, resolve the determinant into the sum of m others, by taking, instead of the first column, each one of the m partial columns; and then, in like manner, resolve each of these into n others, by dealing similarly with the second column. And so, in general, if each of the constituents of a determinant consist of the sum of a number of terms, so that each of the columns can be resolved into the sum of a number of partial columns (the first into m partial columns, the second into n, the third into p, &c.), then the determinant is equal to the sum of all the determinants which can be formed by taking, instead of each column, one of its partial columns; and the number of such determinants will be the product of the numbers m, n, p, &c.

20. *If the constituents of one row or column are respectively equal to the sum of the corresponding constituents of other rows or columns, multiplied respectively by constant factors, the determinant vanishes.* For in this case the determinant can be resolved into the sum of others which separately vanish.

Thus

$$\begin{vmatrix} ka_2 + la_3, & a_2, & a_3 \\ kb_2 + lb_3, & b_2, & b_3 \\ kc_2 + lc_3, & c_2, & c_3 \end{vmatrix} = \begin{vmatrix} ka_2, & a_2, & a_3 \\ kb_2, & b_2, & b_3 \\ kc_2, & c_2, & c_3 \end{vmatrix} + \begin{vmatrix} la_3, & a_2, & a_3 \\ lb_3, & b_2, & b_3 \\ lc_3, & c_2, & c_3 \end{vmatrix}.$$

But the last two determinants vanish (Cor., Art. 15).

21. *A determinant is not altered if we add to each constituent of any row or column the corresponding constituents of any of the other rows or columns multiplied respectively by constant factors.* Thus

$$\begin{vmatrix} a_1 + ka_2 + la_3, & a_2, & a_3 \\ b_1 + kb_2 + lb_3, & b_2, & b_3 \\ c_1 + kc_2 + lc_3, & c_2, & c_3 \end{vmatrix} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 + la_3, & a_2, & a_3 \\ kb_2 + lb_3, & b_2, & b_3 \\ kc_2 + lc_3, & c_2, & c_3 \end{vmatrix}.$$

But the last determinant vanishes (Art. 20).* The following examples will shew how the principles just explained are applied to simplify the calculation of determinants.

Ex. 1. Let it be required to calculate the following determinant:

$$\begin{vmatrix} 9, & 13, & 17, & 4 \\ 18, & 28, & 33, & 8 \\ 30, & 40, & 54, & 13 \\ 24, & 37, & 46, & 11 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1, & 4 \\ 2, & 4, & 1, & 8 \\ 4, & 1, & 2, & 13 \\ 2, & 4, & 2, & 11 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1, & 1 \\ 2, & 4, & 1, & 1 \\ 4, & 1, & 2, & 6 \\ 2, & 4, & 2, & 3 \end{vmatrix}.$$

The second determinant is derived from the first by subtracting from the constituents of the first, second, and third columns, twice, three times, and four times, the corresponding constituents of the last column. The third determinant is derived from the second by subtracting the sum of the first three columns from the last. Whenever we have, as now, a determinant for which all the constituents of one row are equal, we can get by subtraction one for which all the constituents but one of one row vanish, and so reduce the calculation to that of a determinant of lower order (Art. 17). Thus subtracting the first column from each of those following, the determinant last written becomes

$$\begin{vmatrix} 1, & 0, & 0, & 0 \\ 2, & 2, & -1, & -1 \\ 4, & -3, & -2, & 2 \\ 2, & 2, & 0, & 1 \end{vmatrix} = \begin{vmatrix} 2, & -1, & -1 \\ -3, & -2, & 2 \\ 2, & 0, & 1 \end{vmatrix} = \begin{vmatrix} 4, & -1, & -1 \\ -7, & -2, & 2 \\ 0, & 0, & 1 \end{vmatrix} = \begin{vmatrix} 4, & -1 \\ -7, & -2 \end{vmatrix}.$$

The third of these follows from the second by subtracting twice the last column from the first, leaving a determinant of only the second order, whose value is $-8-7=-15$.

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* The beginner will be careful to observe that though the determinant is not altered if we substitute in the first row $a_1 + ka_2 + la_3$ for a_1, &c., yet if we make the same substitution in the second row for a_2, &c., we multiply the determinant by k; and if in the third for a_3, &c., we multiply it by l.

N.B.—It is in all cases possible to make all actual constituents in any row of the same value—by multiplying each and the terms in its column by the product of the others. In this way a determinant can, as in this example, be reduced to one of lower order, and this process is generally the best for numerical calculation.

Ex. 2. The calculation of the following is necessary (*Surfaces*, Art. 234):

$$\begin{vmatrix} 5, & -10, & 11, & 0 \\ -10, & -11, & 12, & 4 \\ 11, & 12, & -11, & 2 \\ 0, & 4, & 2, & -6 \end{vmatrix} = \begin{vmatrix} 5, & -10, & 11, & 0 \\ -32, & -35, & 34, & 0 \\ 11, & 12, & -11, & 2 \\ 1, & 5, & 3, & 0 \end{vmatrix} = -2 \begin{vmatrix} 5, & -10, & 11 \\ -32, & -35, & 34 \\ 1, & 5, & 3 \end{vmatrix}$$

$$= 10 \begin{vmatrix} 5, & -2, & 1 \\ 32, & 7, & 1 \\ 1, & 1, & 8 \end{vmatrix} = 10 \begin{vmatrix} 5, & -2, & 1 \\ 27, & 9, & 0 \\ -39, & 17, & 0 \end{vmatrix} = 90 \begin{vmatrix} 3, & 1 \\ -39, & 17 \end{vmatrix} = 90\,(51 + 39) = 8100.$$

The first transformation is made by subtracting double the third row from the second, and adding the sum of the second and third to the fourth. In the next step it will be observed, that since the sign of the term $a_1b_2c_4d_3$ is opposite to that of $a_1b_2c_3d_4$, when c_4 is the only constituent of the last column which does not vanish, the determinant becomes $-c_4\,(a_1b_2d_3)$. In the next step, we add the second and third columns, we take out the factor 5 common to the second column, and the sign − common to the second row. We then subtract the first row from the second, and eight times the first row from the last, and the remainder is obvious.

Ex. 3.
$$\begin{vmatrix} 7, & -2, & 0, & 5 \\ -2, & 6, & -2, & 2 \\ 0, & -2, & 5, & 3 \\ 5, & 2, & 3, & 4 \end{vmatrix} = -972 \text{ (\textit{Surfaces}, Art. 225).}$$

Ex. 4.
$$\begin{vmatrix} 25, & -15, & 23, & -5 \\ -15, & -10, & 19, & 5 \\ 23, & 19, & -15, & 9 \\ -5, & 5, & 9, & -5 \end{vmatrix} = 194100 \text{ (\textit{Surfaces}, Art. 234).}$$

Ex. 5. Given n quantities α, β, γ, &c., to find the value of

$$\begin{vmatrix} 1, & 1, & 1, & 1, & \&c. \\ \alpha, & \beta, & \gamma, & \delta, & \&c. \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2, & \&c. \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \alpha^{n-1}, & \beta^{n-1}, & \gamma^{n-1}, & \delta^{n-1}, & \&c. \end{vmatrix}.$$

It is evident (Art. 14) that this determinant would vanish if $\alpha = \beta$, therefore $\alpha - \beta$ is a factor in it. In like manner so is every other difference between any two of the quantities α, β, &c. The determinant is therefore

$$= (\beta - \alpha)\,(\gamma - \alpha)\,(\delta - \alpha)\ldots(\gamma - \beta)\,(\delta - \beta)\ldots(\delta - \gamma)\ldots\&c.$$

For the determinant is either equal to this product or to the product multiplied by some factor. But there can be no factor containing α, β, &c., since the product contains α^{n-1}, β^{n-1}, &c.; and the determinant can contain no higher power of α, β, &c.; and by comparing the coefficients of α^{n-1} it will be seen that the determinant contains no numerical factor. This example may also be treated in the same way as the next example.

Ex. 6. To calculate $\begin{vmatrix} 1, & 1, & 1, & 1 \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2 \\ \alpha^4, & \beta^4, & \gamma^4, & \delta^4 \end{vmatrix}$.

Subtract the last column from each of the first three and the determinant becomes divisible by $(\delta-\alpha)\,(\delta-\beta)\,(\delta-\gamma)$, the quotient being

$$\begin{vmatrix} 1, & 1, & 1 \\ \alpha+\delta, & \beta+\delta, & \gamma+\delta \\ \alpha^3+\alpha^2\delta+\alpha\delta^2+\delta^3, & \beta^3+\beta^2\delta+\beta\delta^2+\delta^3, & \gamma^3+\gamma^2\delta+\gamma\delta^2+\delta^3 \end{vmatrix}.$$

Subtract again the last column from the two preceding and the determinant is seen to be divisible by $(\alpha-\gamma)\,(\beta-\gamma)$, and its value is thus at once found to be

$$(\alpha-\delta)\,(\beta-\delta)\,(\gamma-\delta)\,(\alpha-\gamma)\,(\beta-\gamma)\,(\alpha-\beta)\,(\alpha+\beta+\gamma+\delta).*$$

Ex. 7. In the solution of a geometrical problem it became necessary to determine λ from the equation

$$\begin{vmatrix} a^3, & b^3, & c^3 \\ (a+\lambda)^3, & (b+\lambda)^3, & (c+\lambda)^3 \\ (2a+\lambda)^3, & (2b+\lambda)^3, & (2c+\lambda)^3 \end{vmatrix} = 0.$$

Subtract the first row from the second, and divide by λ; subtract 8 times the first row from the last and divide by λ; then subtract the second row from the third and divide by 3; and, lastly, subtract this last row from the second and divide by λ, when the determinant becomes

$$\begin{vmatrix} a^3, & b^3, & c^3 \\ 2a+\lambda, & 2b+\lambda, & 2c+\lambda \\ 3a^2+a\lambda, & 3b^2+b\lambda, & 3c^2+c\lambda \end{vmatrix} = 0.$$

Again, subtract the first column from the second and third, and divide by $b-a$, $c-a$; then subtract the second from the third, and divide by $c-b$; and then from the first column subtract a times the second and add ab times the last; and from the second column take $(a+b)$ times the last, and we have finally

$$\begin{vmatrix} abc, & -(ab+bc+ca), & a+b+c \\ \lambda, & 2 & 0 \\ 0, & \lambda, & 3 \end{vmatrix} = 0,$$

which reduced is

$$(a+b+c)\,\lambda^2+3\,(ab+bc+ca)\,\lambda+6abc=0.$$

Ex. 8. $$\begin{vmatrix} (b+c)^2, & a^2, & a^2 \\ b^2, & (c+a)^2, & b^2 \\ c^2, & c^2, & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$$

Ex. 9. $$\begin{vmatrix} 1, & 1, & 1 \\ \sin\alpha, & \sin\beta, & \sin\gamma \\ \cos\alpha, & \cos\beta, & \cos\gamma \end{vmatrix} = 4\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\gamma).$$

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* On the general theory of which this and the preceding example form part, see note at end on Rational Functional Determinants.

Similarly, by Ex. 5,

$$\begin{vmatrix} \sin^3\alpha, & \sin^2\alpha\cos\alpha, & \sin\alpha\cos^2\alpha, & \cos^3\alpha, \\ \sin^3\beta, & \sin^2\beta\cos\beta, & \sin\beta\cos^2\beta, & \cos^3\beta, \\ \sin^3\gamma, & \sin^2\gamma\cos\gamma, & \sin\gamma\cos^2\gamma, & \cos^3\gamma, \\ \sin^3\delta, & \sin^2\delta\cos\delta, & \sin\delta\cos^2\delta, & \cos^3\delta, \end{vmatrix} =$$

$$\sin(\alpha-\beta)\sin(\alpha-\gamma)\sin(\alpha-\delta)\sin(\beta-\gamma)\sin(\beta-\delta)\sin(\gamma-\delta).$$

Ex. 10. $$\begin{vmatrix} \cos(\alpha-\beta), & \cos(\beta-\gamma), & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta), & \cos(\beta+\gamma), & \cos(\gamma+\alpha) \\ \sin(\alpha+\beta), & \sin(\beta+\gamma), & \sin(\gamma+\alpha) \end{vmatrix} = 2\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\alpha-\gamma).$$

Ex. 11. $$\begin{vmatrix} \sin\alpha, & \cos\alpha, & \sin\alpha\cos\alpha \\ \sin\beta, & \cos\beta, & \sin\beta\cos\beta \\ \sin\gamma, & \cos\gamma, & \sin\gamma\cos\gamma \end{vmatrix}$$

$$= 2\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\{\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)\}.$$

This follows at once from the identities

$$(a'b''+a''b'+a''b+ab''+ab'+a'b)\begin{vmatrix} 1, & a, & b \\ 1, & a', & b' \\ 1, & a'', & b'' \end{vmatrix} =$$

$$\begin{vmatrix} a'b''+a''b'-2ab, & a, & b \\ a''b+ab''-2a'b', & a', & b' \\ ab'+a'b-2a''b'', & a'', & b'' \end{vmatrix} = \begin{vmatrix} 1, & a^2, & b^2 \\ 1, & a'^2, & b'^2 \\ 1, & a''^2, & b''^2 \end{vmatrix} - 2\begin{vmatrix} a, & b, & ab \\ a', & b', & a'b' \\ a'', & b'', & a''b'' \end{vmatrix}.$$

Ex. 12. Many of these examples may be applied to the calculation of areas of triangles, it being remembered that the double area of the triangle formed by three points is

$$\begin{vmatrix} 1, & 1, & 1 \\ x', & x'', & x''' \\ y', & y'', & y''' \end{vmatrix},$$

and by three lines $ax+by+c$, &c., is

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}^2 \text{ divided by } (a'b''-a''b')(a''b-ab'')(ab'-a'b)$$

(see *Conic Sections*, p. 32). For example, the area of the triangle formed by the centres of curvature of three points on a parabola is (the coordinates of a centre of curvature being $\frac{1}{2}p+3x, -\frac{4y^3}{p^2}$)

$$\frac{6}{p^3}\begin{vmatrix} 1, & 1, & 1 \\ y^2, & y'^2, & y''^2 \\ y^3, & y'^3, & y''^3 \end{vmatrix} = \frac{6}{p^3}(y'-y'')(y''-y)(y-y')(y'y''+y''y+yy').$$

In like manner may be investigated the area of the triangle formed by three normals, or any other three lines connected with the curve.

Ex. 13. $$\begin{vmatrix} 0, & c, & b \\ c, & 0, & a \\ b, & a, & 0 \end{vmatrix} = 2abc; \qquad \begin{vmatrix} 0, & c, & b & d \\ c, & 0, & a, & e \\ b, & a, & 0, & f \\ d, & e, & f, & 0 \end{vmatrix} = a^2d^2+b^2e^2+c^2f^2-2abde-2bcef-2adcf.$$

Ex. 14. Prove $\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & z^2, & y^2 \\ 1, & z^2, & 0, & x^2 \\ 1, & y^2, & x^2, & 0 \end{vmatrix} = \begin{vmatrix} 0, & x, & y, & z \\ x, & 0, & z, & y \\ y, & z, & 0, & x \\ z, & y, & x, & 0 \end{vmatrix}$ and $\begin{vmatrix} t, & x, & y, & z \\ x, & t, & z, & y \\ y, & z, & t, & x \\ z, & y, & x, & t \end{vmatrix}$

$$= (t+x+y+z)\,(t+x-y-z)\,(t-x+y-z)\,(t-x-y+z).$$

Ex. 15. $$\begin{vmatrix} a, & \lambda, & \lambda, & \lambda, & \&c. \\ \lambda, & b, & \lambda, & \lambda, & \&c. \\ \lambda, & \lambda, & c, & \lambda, & \&c. \\ \lambda, & \lambda, & \lambda, & d, & \&c. \\ & & \&c. & & \end{vmatrix},$$

where all the constituents are equal except those in the principal diagonal, is $\phi(\lambda) - \lambda \dfrac{d\phi}{d\lambda}$; where $\phi(\lambda)$ is the continued product $(a-\lambda)\,(b-\lambda)$ &c.

Ex. 16. Let u be a homogeneous function of the n^{th} order in any number of variables; and let u_1, u_2, u_3, &c., denote its differential coefficients with regard to the variables x_1, x_2, x_3, &c.; and in like manner, let u_{11}, u_{12}, u_{13} denote the differential coefficients of u_1, &c. Then, by Euler's theorem of homogeneous functions, we have

$$nu = u_1x_1 + u_2x_2 + u_3x_3 + \&c., \quad (n-1)\,u_1 = x_1u_{11} + x_2u_{12} + x_3u_{13} + \&c., \quad \&c.$$

We shall hereafter speak at length of the determinant (called the Hessian) formed with the second differential coefficients, whose rows are u_{11}, u_{12}, u_{13}, &c.; u_{21}, u_{22}, u_{23}, &c., &c. At present our object is to shew how to reduce a class of determinants of frequent occurrence, which are formed by bordering the Hessian, either with the first differential coefficients, or with other quantities, as for example

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_1, & a_1 \\ u_{21}, & u_{22}, & u_{23}, & u_2, & a_2 \\ u_{31}, & u_{32}, & u_{33}, & u_3, & a_3 \\ u_1, & u_2, & u_3, & 0, & 0 \\ a_1, & a_2, & a_3 & 0, & 0, \end{vmatrix}.$$

In this example we only take three variables, and the determinant formed by the first three rows and columns is the Hessian, which we shall call H, but the processes which we shall employ are applicable to the case of any number of variables. We denote the above determinant by the abbreviation $\begin{pmatrix} u & \alpha \\ u & \alpha \end{pmatrix}$, and use $\begin{pmatrix} u \\ u \end{pmatrix}$, $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$, $\begin{pmatrix} u \\ \alpha \end{pmatrix}$ to denote the determinants of four rows, formed by bordering the Hessian with a single row and column, either both u's or both α's, or one u and the other α. We also write $\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 = \alpha$. If now we multiply the first column of the above written determinant by x_1, the second by x_2, the third by x_3, and subtract the sum from $n-1$ times the fourth column, the first three terms vanish, the fourth becomes $-nu$, and the fifth $-\alpha$. Again, multiply the fourth row by $n-1$, and subtract in like manner the sum of the first, second, and third rows multiplied by x_1, x_2, x_3 respectively, then the first three terms vanish, the fourth remains unchanged, and the last becomes $-\alpha$. Thus then $(n-1)^2$ times the determinant originally

written is proved to be equal to

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & 0, & \alpha_1 \\ u_{21}, & u_{22}, & u_{23}, & 0, & \alpha_2 \\ u_{31}, & u_{32}, & u_{33}, & 0, & \alpha_3 \\ 0, & 0, & 0, & -n(n-1)u, & -\alpha \\ \alpha_1, & \alpha_2, & \alpha_3, & -\alpha, & 0 \end{vmatrix}.$$

But now since (Art. 15) a determinant which has only two terms d_4, d_5 of the fourth row which do not vanish, is expressible in the form $d_4D_4 + d_5D_5$; the above determinant may be resolved into the sum of two others, and we find that the originally given determinant

$$\begin{pmatrix} u & \alpha \\ u & \alpha \end{pmatrix} = -\frac{n}{n-1} u \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} - \frac{1}{(n-1)^2} \alpha^2 H.$$

In like manner it is proved that $\begin{pmatrix} u \\ u \end{pmatrix} = -\frac{n}{n-1} Hu$. Or, again, if there be four variables, and the Hessian be triply bordered, we prove in the same way that

$$\begin{pmatrix} u & \alpha & \beta \\ u & \alpha & \beta \end{pmatrix} = -\frac{n}{n-1} u \begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix} - \frac{1}{(n-1)^2} \alpha^2 \begin{pmatrix} \beta \\ \beta \end{pmatrix} - \frac{1}{(n-1)^2} \beta^2 \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} + \frac{2}{(n-1)^2} \alpha\beta \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

When u is of the second degree, it is to be noted that, in the case of three variables, $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = 0$ is the condition that the line α should touch the conic u; and $\begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix} = 0$ is the condition that the intersection of the lines α, β should be on the conic. In like manner for four variables the vanishing of $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$, $\begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}$, $\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix}$ respectively, are the conditions that a plane should touch, that the intersection of two planes should touch, and that the intersection of three planes should be on, the quadric u. The equation then $\begin{pmatrix} u & \alpha & \beta \\ u & \alpha & \beta \end{pmatrix} = 0$ expresses that the polar plane of a point passes through one or other of the two points where the line $\alpha\beta$ meets the quadric. But points having this property lie only on the tangent planes at these two points. The transformation, therefore, that we have given for $\begin{pmatrix} u & \alpha & \beta \\ u & \alpha & \beta \end{pmatrix}$ expresses the equation of the tangent planes at the points where $\alpha\beta$ meets the quadric, and the transformation for $\begin{pmatrix} u & \alpha \\ u & \alpha \end{pmatrix}$ gives the equation of the tangent cone where α meets the quadric.

Ex. 17. Find the value of a determinant of the form

$$\begin{vmatrix} a, & 1, & 0, & 0, & 0 \\ -1, & b, & 1, & 0, & 0 \\ 0, & -1, & c, & 1, & 0 \\ 0, & 0, & -1, & d, & 1 \\ 0, & 0, & 0, & -1, & e \end{vmatrix}.$$

Determinants in which all the constituents vanish except those in the principal diagonal and the two bordering minor diagonals, have been studied by Mr. Muir under the name of continuants (*Proceedings of the Royal Soc., Edinb.*, 1873–4). The above determinant may be written in the abbreviated form (a, b, c, d, e); and taking out the constituents in the first row (as in Art. 16), the value of the determinant is seen to be $a(b, c, d, e) + (c, d, e)$. In this way we can easily form the series of values of continuants of two, three, &c. rows, viz.

$$ab + 1, \quad abc + c + a, \quad abcd + cd + ad + ab + 1,$$
$$abcde + abc + abe + ade + cde + a + c + e, \text{ &c.}$$

The rule of formation is, take the product *abcde* of all the constituents, and omit from it in every possible way the pairs of consecutive literal constituents. Thus, in the last case, the omitted pairs are *de*, *cd*, *bc*, *ab*, (*bc*, *de*), (*ab*, *de*), (*ab*, *cd*).

Determinants of the class here described occur in the theory of continued fractions; for it is obvious that the successive approximations to the value of the continued fraction $a + \frac{1}{b+} \frac{1}{c+}$, &c., are

$$(a), \frac{(a, b)}{(b)}, \frac{(a, b, c)}{(b, c)}, \frac{(a, b, c, d)}{(b, c, d)}, \text{ &c.}$$

Ex. 18. Find the number of terms in a continuant of the n^{th} order. From the equation $(a, b, c, d, e) = a\,(b, c, d, e) + (c, d, e)$, it is obvious that if (n) be the number required, we have the relation $(n) = (n-1) + (n-2)$; and that therefore for the orders 1, 2, 3, &c., we have the series of numbers 1, 2, 3, 5, 8, 13, &c.; and generally if $\left\{\frac{1+\sqrt{(5)}}{2}\right\}^n = A_n + B_n \sqrt{(5)}$, the number required is $A_n + B_n$.

LESSON III.

MULTIPLICATION OF DETERMINANTS.

22. THE product of two determinants may be at once written down as a determinant whose order is the sum of their orders. For instance, the product of $(a_1 b_2)$, $(\alpha_1 \beta_2 \gamma_3)$ may be written down

$$\begin{vmatrix} a_1, & a_2, & 0, & 0, & 0 \\ b_1, & b_2, & 0, & 0, & 0 \\ 0, & 0, & \alpha_1, & \alpha_2, & \alpha_3 \\ 0, & 0, & \beta_1, & \beta_2, & \beta_3 \\ 0, & 0, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix},$$

as is evident on expanding the last written determinant. (See also *post* Art. 32). We shall now shew that the product of two determinants may be written down as a determinant whose order does not exceed the highest of the two.

The product of two determinants of the same order is the determinant whose constituents are the sums of the products of the constituents in any row of one by the corresponding constituents in any row of the other.

For example, the product of the determinants $(a_1b_2c_3)$ and $(\alpha_1\beta_2\gamma_3)$ is

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1, & a_1\alpha_2+b_1\beta_2+c_1\gamma_2, & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1, & a_2\alpha_2+b_2\beta_2+c_2\gamma_2, & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1, & a_3\alpha_2+b_3\beta_2+c_3\gamma_2, & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}.$$

The proofs which we shall give for this particular case will apply equally in general. Since the constituents of the determinant just written are each the sum of three terms, the determinant can (by Art. 19) be resolved into the sum of the 27 determinants, obtained by taking any one partial column of the first, second, and third columns. We need not write down the whole 27, but give two or three specimen terms:

$$\begin{vmatrix} a_1\alpha_1, & a_1\alpha_2, & a_1\alpha_3 \\ a_2\alpha_1, & a_2\alpha_2, & a_2\alpha_3 \\ a_3\alpha_1, & a_3\alpha_2, & a_3\alpha_3 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1, & b_1\beta_2, & c_1\gamma_3 \\ a_2\alpha_1, & b_2\beta_2, & c_2\gamma_3 \\ a_3\alpha_1, & b_3\beta_2, & c_3\gamma_3 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1, & c_1\gamma_2, & b_1\beta_3 \\ a_2\alpha_1, & c_2\gamma_2, & b_2\beta_3 \\ a_3\alpha_1, & c_3\gamma_2, & b_3\beta_3 \end{vmatrix} + \&c.$$

Now it will be observed that in all these determinants each column has a common factor, which (Art. 15) may be taken out as a multiplier of the entire determinant. The specimen terms already given may therefore be written in the form

$$\alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1, & a_1, & a_1 \\ a_2, & a_2, & a_2 \\ a_3, & a_3, & a_3 \end{vmatrix} + \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} + \alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1, & c_1, & b_1 \\ a_2, & c_2, & b_2 \\ a_3, & c_3, & b_3 \end{vmatrix}.$$

But the first of these determinants vanishes, since two columns are the same; the second is the determinant $(a_1b_2c_3)$; and the third (Art. 13) is $=-(a_1b_2c_3)$. In like manner, every other partial determinant will vanish which has two columns the same; and it will be found that every determinant which does not vanish will be $(a_1b_2c_3)$, while the factors which multiply it will be the elements of the determinant $(\alpha_1\beta_2\gamma_3)$.

It would have been equally possible to break up the determinant into a series of terms, every one of which would have been the determinant $(\alpha_1\beta_2\gamma_3)$ multiplied by one of the elements of $(a_1b_2c_3)$.

The theorem of this article is applicable to the multiplication of determinants of different orders, because it has been shewn (Art. 17) that we can always write that of the lower order as one of the same order with the higher.

23. On account of the importance of this theorem, we give another proof, founded on our first definition of a determinant.

The determinant which we examined in the last Article is the result of elimination between the equations

$$(a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1)x + (a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2)y + (a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3)z = 0,$$
$$(a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1)x + (a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2)y + (a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3)z = 0,$$
$$(a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1)x + (a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2)y + (a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3)z = 0.$$

Now if we write

$$\alpha_1 x + \alpha_2 y + \alpha_3 z = X,$$
$$\beta_1 x + \beta_2 y + \beta_3 z = Y,$$
$$\gamma_1 x + \gamma_2 y + \gamma_3 z = Z,$$

the three preceding equations may be written

$$a_1 X + b_1 Y + c_1 Z = 0,$$
$$a_2 X + b_2 Y + c_2 Z = 0,$$
$$a_3 X + b_3 Y + c_3 Z = 0,$$

from which, eliminating X, Y, Z, we see at once that $(a_1 b_2 c_3)$ must be a factor in the result. But also a system of values of x, y, z can be found to satisfy the three given equations, provided a system can be found to satisfy simultaneously the equations $X = 0$, $Y = 0$, $Z = 0$. Hence $(\alpha_1 \beta_2 \gamma_3) = 0$, which is the condition that the latter should be possible, is also a factor in the result. And since we can see without difficulty that the degree of the result in the coefficients is exactly the same as that of the product of these quantities, the result is $(a_1 b_2 c_3)(\alpha_1 \beta_2 \gamma_3)$.

So in general the product of any number of determinants may be expressed as a determinant whose order does not exceed the highest of the orders of the given determinants, and whose elements are rational functions of the elements of the given determinants.

It appears from the present Article that the theorem concerning the multiplication of determinants can be expressed in the following form, in which we shall frequently employ it: *If a system of equations*

$$a_1 X + b_1 Y + c_1 Z = 0,\ a_2 X + b_2 Y + c_2 Z = 0,\ a_3 X + b_3 Y + c_3 Z = 0$$

be transformed by the substitutions

$$X=\alpha_1 x+\alpha_2 y+\alpha_3 z,\quad Y=\beta_1 x+\beta_2 y+\beta_3 z,\quad Z=\gamma_1 x+\gamma_2 y+\gamma_3 z,$$

then the determinant of the transformed system will be equal to $(a_1b_2c_3)$ *the determinant of the original system, multiplied by* $(\alpha_1\beta_2\gamma_3)$, which we shall call *the modulus of transformation.*

24. The theorems of the last Articles may be extended as follows: We might have two sets of constituents, the number of rows being different from the number of columns; for example

$$\left\| \begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{matrix} \right\| \quad \left\| \begin{matrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{matrix} \right\|,$$

and from these we could form, in the same manner as in the last Articles, the determinant

$$\left| \begin{matrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1, & a_2\alpha_1+b_2\beta_1+c_2\gamma_1 \\ a_1\alpha_2+b_1\beta_2+c_1\gamma_2, & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 \end{matrix} \right|,$$

whose value we purpose to investigate.

Now, first, let the number of columns be greater than the number of rows, as in the example just written, so that each constituent of the new determinant is the sum of a number of terms greater than the number of rows; then proceeding, as in Art. 22, the value of the determinant is

$$\left| \begin{matrix} a_1\alpha_1, & a_2\alpha_1 \\ a_1\alpha_2, & a_2\alpha_2 \end{matrix} \right| + \left| \begin{matrix} a_1\alpha_1, & b_2\beta_1 \\ a_1\alpha_2, & b_2\beta_2 \end{matrix} \right| + \&c.$$

$$= (b_1c_2)\,(\beta_1\gamma_2) + (c_1a_2)\,(\gamma_1\alpha_2) + (a_1b_2)\,(\alpha_1\beta_2).$$

That is to say, *the new determinant is the sum of the products of every possible determinant which can be formed out of the one set of constituents by the corresponding determinant formed out of the other set of constituents.*

25. But in the second place, let the number of rows exceed the number of columns. Thus, from the two sets of constituents,

$$\left\| \begin{matrix} a_1, & b_1 \\ a_2, & b_2 \\ a_3, & b_3 \end{matrix} \right\| \quad \left\| \begin{matrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \\ \alpha_3, & \beta_3 \end{matrix} \right\|,$$

let us form the determinant

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1, & a_2\alpha_1+b_2\beta_1, & a_3\alpha_1+b_3\beta_1 \\ a_1\alpha_2+b_1\beta_2, & a_2\alpha_2+b_2\beta_2, & a_3\alpha_2+b_3\beta_2 \\ a_1\alpha_3+b_1\beta_3, & a_2\alpha_3+b_2\beta_3, & a_3\alpha_3+b_3\beta_3 \end{vmatrix}.$$

Then when we proceed to break this up into partial determinants in the manner already explained, it will be found impossible to form any partial determinant which shall not have two columns the same. *The determinant, therefore, will vanish identically.* Or this may be seen immediately by adding a column of cyphers to each matrix and then multiplying, when we get the determinant last written as the product of two factors each equal zero.

26. A useful particular case of Art. 22 is, that *the square of a determinant is a symmetrical determinant* (see Art. 10). Thus the square of $(a_1b_2c_3)$ is

$$\begin{vmatrix} a_1^2+b_1^2+c_1^2, & a_1a_2+b_1b_2+c_1c_2, & a_1a_3+b_1b_3+c_1c_3 \\ a_1a_2+b_1b_2+c_1c_2, & a_2^2+b_2^2+c_2^2, & a_2a_3+b_2b_3+c_2c_3 \\ a_1a_3+b_1b_3+c_1c_3, & a_2a_3+b_2b_3+c_2c_3, & a_3^2+b_3^2+c_3^2 \end{vmatrix}.$$

Again, it appears by Art. 24, that the sum of the squares of the determinants $(b_1c_2)^2+(c_1a_2)^2+(a_1b_2)^2$ is the determinant

$$\begin{vmatrix} a_1^2+b_1^2+c_1^2, & a_1a_2+b_1b_2+c_1c_2 \\ a_1a_2+b_1b_2+c_1c_2, & a_2^2+b_2^2+c_2^2 \end{vmatrix}.$$

Ex. 1. If a_1, b_1, c_1; a_2, b_2, c_2 be the direction-cosines of two lines in space, and θ their inclination to each other, $\cos\theta = a_1a_2 + b_1b_2 + c_1c_2$; and the identity last proved gives $\sin^2\theta = (b_1c_2)^2 + (c_1a_2)^2 + (a_1b_2)^2$.

Ex. 2. In the theory of equations it is important to express the product of the squares of the differences of the roots; now the product of the differences of n quantities has been expressed as a determinant (Ex. 5, p. 14), and if we form the square of this determinant we obtain

$$\begin{vmatrix} s_0, & s_1, & s_2 \ldots s_{n-1} \\ s_1, & s_2, & s_3 \ldots s_n \\ s_2, & s_3, & s_4 \ldots s_{n+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ s_{n-1}, & s_n, & s_{n+1} \ldots s_{2n-2} \end{vmatrix},$$

where s_p denotes the sum of the p^{th} powers of the quantities α, β, &c.

Ex. 3. In like manner it is proved, by Art. 24, that the determinant

$$\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix} = \Sigma\,(\alpha-\beta)^2,$$

$$\begin{vmatrix} s_0, & s_1, & s_2 \\ s_1, & s_2, & s_3 \\ s_2, & s_3, & s_4 \end{vmatrix} = \Sigma\,(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\alpha-\beta)^2.$$

We thus form a series of determinants, the last of which is the product of the squares of the differences of α, β, &c.; all similar determinants beyond this vanish identically by Art. 25. This series of determinants is of great importance in the theory of algebraic equations.

Ex. 4. If there be any n quantities α, β, γ, &c., and a like number α', β', γ', &c., the product may be formed

$$\begin{vmatrix} \alpha^r, & r\alpha^{r-1}, & \frac{1}{2}r(r-1)\,\alpha^{r-2}\ldots 1 \\ \beta^r, & r\beta^{r-1}, & \frac{1}{2}r(r-1)\,\beta^{r-2}\ldots 1 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} \begin{vmatrix} 1-\alpha', & \alpha'^2\ldots(-\alpha')^r \\ 1-\beta', & \beta'^2\ldots(-\beta')^r \\ \ldots\ldots & \ldots\ldots \end{vmatrix}$$

$$= \begin{vmatrix} (\alpha-\alpha')^r, & (\alpha-\alpha')^r, & (\alpha-\gamma')^r\ldots \\ (\beta-\alpha')^r, & (\beta-\beta')^r, & (\beta-\gamma')^r\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix},$$

when if the number $n > r+1$ each side vanishes; otherwise as each determinant of the first matrix contains as factors every difference of the first quantities, and each of the second every difference of the second set, the resulting determinant is divisible by the product of both continued products of differences.

Ex. 5. Various well-known identities may be established by the method of Art. 25. Thus it is easily seen that

$$\begin{vmatrix} \cos\alpha, & \sin\alpha \\ \cos\beta, & \sin\beta \\ \cos\gamma, & \sin\gamma \end{vmatrix} \begin{vmatrix} \cos\alpha, & \sin\alpha \\ \cos\beta, & \sin\beta \\ \cos\gamma, & \sin\gamma \end{vmatrix},$$

gives

$$\begin{vmatrix} 1, & \cos(\beta-\alpha), & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta), & 1, & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma), & \cos(\beta-\gamma), & 1 \end{vmatrix} = 0,$$

similarly

$$\begin{vmatrix} \cos\alpha, & \sin\alpha \\ \cos\beta, & \sin\beta \\ \cos\gamma, & \sin\gamma \end{vmatrix} \begin{vmatrix} \sin\alpha', & \cos\alpha' \\ \sin\beta', & \cos\beta' \\ \sin\gamma', & \cos\gamma' \end{vmatrix} = \begin{vmatrix} \sin(\alpha+\alpha'), & \sin(\beta+\alpha'), & \sin(\gamma+\alpha') \\ \sin(\alpha+\beta'), & \sin(\beta+\beta'), & \sin(\gamma+\beta') \\ \sin(\alpha+\gamma'), & \sin(\beta+\gamma'), & \sin(\gamma+\gamma') \end{vmatrix} = 0.$$

If a conic break up into two lines $(ax+\beta y+\gamma z)$, $(a'x+\beta'y+\gamma'z)$, we find, equating coefficients,

$$\begin{vmatrix} \alpha, & \alpha' \\ \beta, & \beta' \\ \gamma, & \gamma' \end{vmatrix} \begin{vmatrix} \alpha', & \alpha \\ \beta', & \beta \\ \gamma', & \gamma \end{vmatrix} = \begin{vmatrix} 2a, & 2h, & 2g \\ 2h, & 2b, & 2f \\ 2g, & 2f, & 2c \end{vmatrix} = 0 \quad \text{(cf. } \textit{Conics}\text{, Art. 77).}$$

If a biquadratic be of the form $(ax+\beta)^4+(a'x+\beta')^4$, by similarly identifying

$$\begin{vmatrix} \alpha^2, & \alpha'^2 \\ \alpha\beta, & \alpha'\beta' \\ \beta^2, & \beta'^2 \end{vmatrix} \begin{vmatrix} \alpha^2, & \alpha'^2 \\ \alpha\beta, & \alpha'\beta' \\ \beta^2, & \beta'^2 \end{vmatrix} = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0.$$

Ex. 6. In like manner, from Art. 24, the two methods of writing

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \end{vmatrix} \begin{vmatrix} c, & a \\ -2b, & a' \\ c', & \end{vmatrix} - 2b' = \begin{vmatrix} 2(ac-b^2) & ac'+a'c-2bb' \\ ac'+a'c-2bb', & 2(a'c'-b'^2) \end{vmatrix}$$

$$= 2(bc')(ab') - (ca')^2 + 2(ab')(bc'),$$

give the identity

$$4(ac-b^2)(a'c'-b'^2) - (ac'+a'c-2bb')^2 = 4(bc')(ab') - (ca')^2.$$

Again,

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \end{vmatrix}^2 = \begin{vmatrix} a^2+b^2+c^2+d^2, & aa'+bb'+cc'+dd', \\ aa'+bb'+cc'+dd', & a'^2+b'^2+c'^2+d'^2 \end{vmatrix} = (bc')^2+(ca')^2+(ab')^2 + (ad')^2+(bd')^2+(cd')^2;$$

but $$(bc')\ (ad')+(ca')\ (bd')+(ab')\ (cd') = \begin{vmatrix} ad'-a'd, & a, & a' \\ bd'-b'd, & b, & b' \\ cd'-c'd, & c, & c' \end{vmatrix} = 0,$$

hence the right side may be written

$$\{(bc')+(ad')\}^2+\{(ca')+(bd')\}^2+\{(ab')+(cd')\}^2,$$

and the identity exhibits the product

$$(a^2+b^2+c^2+d^2)\ (a'^2+b'^2+c'^2+d'^2),$$

as the sum of four squares, a theorem due to Euler.

Ex. 7. The relation among six distances from a point along a line to points which form an involution is that the determinant

$$\begin{vmatrix} 1, & a_1+a_2, & a_1a_2 \\ 1, & b_1+b_2, & b_1b_2 \\ 1, & c_1+c_2, & c_1c_2 \end{vmatrix}$$

may vanish. (*Conics*, Note, p. 310).

Multiplying this by $$\begin{vmatrix} x^2, & -x, & 1 \\ y^2, & -y, & 1 \\ z^2, & -z, & 1 \end{vmatrix},$$

we get $$\begin{vmatrix} (x-a_1)\ (x-a_2), & (y-a_1)\ (y-a_2), & (z-a_1)\ (z-a_2) \\ (x-b_1)\ (x-b_2), & (y-b_1)\ (y-b_2), & (z-b_1)\ (z-b_2) \\ (x-c_1)\ (x-c_2), & (y-c_1)\ (y-c_2), & (z-c_1)\ (z-c_2) \end{vmatrix},$$

by taking the values a_1 for x, b_1 for y, c_1 for z, this reduces to

$$\begin{vmatrix} 0, & (b_1-a_1)\ (b_1-a_2), & (c_1-a_1)\ (c_1-a_2) \\ (a_1-b_1)\ (a_1-b_2), & 0, & (c_1-b_1)\ (c_1-b_2) \\ (a_1-c_1)\ (a_1-c_2), & (b_1-c_1)\ (b_1-c_2), & 0 \end{vmatrix}$$

$$= (b_1-c_1)\ (c_1-a_1)\ (a_1-b_1)\ \{(a_1-b_2)\ (b_1-c_2)\ (c_1-a_2)+(a_2-b_1)\ (b_2-c_1)\ (c_2-a_1)\},$$

whence the original determinant is expressed as the sum of the last two products, or as

$$= \begin{vmatrix} 0, & a_2-b_1, & c_1-a_2 \\ a_1-b_2, & 0, & b_2-c_1 \\ c_2-a_1, & b_1-c_2, & 0 \end{vmatrix}.$$

Ex. 8. Let the origin be taken at the centre of the circle circumscribing a triangle, whose radius is R; and let M be the area of the triangle, then

$$2MR = \begin{vmatrix} x', & y', & R \\ x'', & y'', & R \\ x''', & y''', & R \end{vmatrix} \text{ and } -2MR = \begin{vmatrix} x', & y', & -R \\ x'', & y'', & -R \\ x''', & y''', & -R \end{vmatrix}.$$

Multiply these determinants according to the rule, and the first term $x'^2+y'^2-R^2$ vanishes; the second $x'x''+y'y''-R^2 = -\frac{1}{2}\{(x'-x'')^2+(y'-y'')^2\} = -\frac{1}{2}c^2$, where c is a side of the triangle. Hence then

$$-4M^2R^2 = -\frac{1}{8}\begin{vmatrix} 0, & c^2, & b^2 \\ c^2, & 0, & a^2 \\ b^2, & a^2, & 0 \end{vmatrix} = -\frac{1}{4}a^2b^2c^2,$$

whence $R = \dfrac{abc}{4M}$, as is well known.

Ex. 9. The same process may be used to find an expression for the radius of the sphere circumscribing a tetrahedron. Starting with the expression for the volume of the tetrahedron

$$6V = \begin{vmatrix} x', & y', & z', & 1 \\ x'', & y'', & z'', & 1 \\ x''', & y''', & x''', & 1 \\ x^{iv}, & y^{iv}, & z^{iv}, & 1 \end{vmatrix}.$$

We find, as before, if a, d; b, e; c, f are pairs of opposite edges of the tetrahedron

$$-36R^2V^2 = \tfrac{1}{16} \begin{vmatrix} 0, & c^2, & b^2, & d^2 \\ c^2, & 0, & a^2, & e^2 \\ b^2, & a^2, & 0, & f^2 \\ d^2, & e^2, & f^2, & 0 \end{vmatrix},$$

whence if $ad + be + cf = 2S$; by Ex. 13, p. 16, we find

$$36R^2V^2 = S\,(S-ad)\,(S-be)\,(S-cf).$$

Ex. 10. The above proofs by Mr. Burnside were suggested by the following proof by Joachimsthal of an expression for the area of a triangle inscribed in an ellipse. Multiply the equations

$$\frac{2M}{ab} = \begin{vmatrix} \frac{x'}{a}, & \frac{y'}{b}, & 1 \\ \frac{x''}{a}, & \frac{y''}{b}, & 1 \\ \frac{x'''}{a}, & \frac{y'''}{b}, & 1 \end{vmatrix}; \quad -\frac{2M}{ab} = \begin{vmatrix} \frac{x'}{a}, & \frac{y'}{b}, & -1 \\ \frac{x''}{a}, & \frac{y''}{b}, & -1 \\ \frac{x'''}{a}, & \frac{y'''}{b}, & -1 \end{vmatrix}.$$

And the product is a symmetrical determinant, of which the leading terms, such as $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1$, vanish when the points $x'y'$, $x''y''$, $x'''y'''$ are on the curve, while the other terms are $\frac{x'x''}{a^2} + \frac{y'y''}{b^2} - 1$, &c. Now it can easily be proved that if γ be a side of the triangle, and b''' the parallel semi-diameter,

$$\frac{\gamma^2}{b'''^2} = \frac{(x'-x'')^2}{a^2} + \frac{(y'-y'')^2}{b^2} = 2\left(1 - \frac{x'x''}{a^2} - \frac{y'y''}{b^2}\right).$$

Thus we have

$$-4\frac{M^2}{a^2b^2} = -\tfrac{1}{8} \begin{vmatrix} 0, & \frac{\gamma^2}{b'''^2}, & \frac{\beta^2}{b''^2} \\ \frac{\gamma^2}{b'''^2}, & 0, & \frac{\alpha^2}{b'^2} \\ \frac{\beta^2}{b''^2}, & \frac{\alpha^2}{b'^2}, & 0 \end{vmatrix} = -\frac{\alpha^2\beta^2\gamma^2}{4b'^2b''^2b'''^2}.$$

Ex. 11. The following investigation of the relation connecting the mutual distances of four points on a circle (or five points on a sphere) was given by Prof. Cayley (*Cambridge and Dublin Mathematical Journal*, vol. II., p. 270).

Substitute the coordinates of each point in the general equation of a circle

$$x^2 + y^2 - 2Ax - 2By + C = 0,$$

and eliminate A, B, C, when we get a determinant with four rows such as

$$x'^2 + y'^2, \quad -2x', \quad -2y', \quad 1.$$

Multiply this by the determinant (which only differs by a numerical factor from the

preceding) whose four rows are such as 1, x', y', $x'^2+y'^2$, and the first term of the product determinant vanishes, the second being $(x'-x'')^2+(y'-y'')^2$. If then the square of the distance between two of the points be $(12)^2$, the product determinant is

$$\begin{vmatrix} 0, & (12)^2, & (13)^2, & (14)^2 \\ (21)^2, & 0, & (23)^2, & (24)^2 \\ (31)^2, & (32)^2, & 0, & (34)^2 \\ (41)^2, & (42)^2, & (43)^2 & 0 \end{vmatrix} = 0,$$

which is the relation required. As has been already seen, this determinant expanded gives the well-known relation $(12)(34) \pm (13)(24) \pm (14)(23) = 0$. The relation connecting five points on a sphere is the corresponding determinant with five rows.

Ex. 12. To find a relation connecting the mutual distances of three points on a line, four points on a plane, or five points in space. We prefix a unit and cyphers to the two determinants which we multiplied in the last example, thus

$$\begin{vmatrix} 1, & 0, & 0, & 0 \\ x'^2+y'^2, & -2x', & -2y', & 1 \\ & \&c. & & \end{vmatrix} \times \begin{vmatrix} 0, & 0, & 0, & 1 \\ 1, & x', & y', & x'^2+y'^2 \\ & \&c. & & \end{vmatrix}.$$

We have now got five rows and only four columns, therefore the product formed, as in Art. 25, will vanish identically. But this is the determinant

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & (12)^2, & (13)^2, & (14)^2 \\ 1, & (21)^2, & 0, & (23)^2, & (24)^2 \\ 1, & (31)^2, & (32)^2, & 0, & (34)^2 \\ 1, & (41)^2, & (42)^2, & (43)^2, & 0 \end{vmatrix} = 0,$$

which is the relation required. If we erase the outside row and column, we have the relation connecting three points on a line; and if we add another row, 1, $(51)^2$, $(52)^2$, &c., and symmetrical column, we get the relation connecting the mutual distances of five points in space. We might proceed to calculate these determinants by subtracting the second column from each of those succeeding, and then the first row from those succeeding, when we get

$$\begin{vmatrix} 2(12)^2, & (12)^2+(13)^2-(23)^2, & (12)^2+(14)^2-(24)^2 \\ (12)^2+(13)^2-(23)^2, & 2(13)^2, & (13)^2+(14)^2-(34)^2 \\ (12)^2+(14)^2-(24)^2, & (13)^2+(14)^2-(34)^2, & 2(14)^2 \end{vmatrix} = 0.$$

Now the determinants might have been obtained directly in this reduced but less symmetrical form by taking the origin at the point (1), and forming, as in Art. 25, with the constituents $x'y'$, $x''y''$, &c., the determinant which vanishes identically,

$$\begin{vmatrix} x'^2+y'^2, & x'x''+y'y'', & x'x'''+y'y''' \\ x'x''+y'y'', & x''^2+y''^2, & x''x'''+y''y''' \\ x'x'''+y'y''', & x''x'''+y''y''', & x'''^2+y'''^2 \end{vmatrix},$$

which it will readily be seen is equivalent to that last written.

Ex. 13. To find the relation connecting the arcs which join four points on a sphere. Take the origin at the centre of the sphere, and form with the direction-cosines of the radii vectores to each point, $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$; $\cos\alpha''$, &c., a determinant which vanishes identically, this will be

$$\begin{vmatrix} 1, & \cos ab, & \cos ac, & \cos ad \\ \cos ba, & 1, & \cos bc, & \cos bd \\ \cos ca, & \cos cb, & 1, & \cos cd \\ \cos da, & \cos db, & \cos dc, & 1 \end{vmatrix} = 0.$$

If we substitute for each cosine, $\cos ab$, $1 - \frac{(ab)^2}{2r^2}$ + &c., and then suppose r the radius of the sphere to be infinite, we derive from the determinant of this article the relation of last article connecting four points on a plane.

Ex. 14. If $\phi(\lambda) = \begin{vmatrix} a-\lambda, & h, & g, \\ h, & b-\lambda, & f, \\ g, & f, & c-\lambda, \end{vmatrix}$, calculate $\phi(\lambda).\phi(-\lambda)$. The new determinant is one of like form with λ^2 instead of λ, the first line being $A-\lambda^2$, H, G, &c., where

$$A = a^2 + h^2 + g^2, \quad B = b^2 + f^2 + h^2, \quad C = c^2 + g^2 + f^2,$$

$$F = gh + f(b+c), \quad G = hf + g(c+a), \quad H = fg + h(a+b),$$

and the expanded determinant equated to cypher gives $\lambda^6 - L\lambda^4 + M\lambda^2 - N = 0$, where

$$L = a^2 + b^2 + c^2 + 2(f^2 + g^2 + h^2),$$

$$M = (bc - f^2)^2 + (ca - g^2)^2 + (ab - h^2)^2 + 2(af - gh)^2 + 2(bg - hf)^2 + 2(ch - fg)^2,$$

and N is the square of the original determinant with λ in it $= 0$; L, M, N are then all essentially positive quantities. In like manner if $\phi(\lambda)$ be formed similarly from any symmetrical determinant, $\phi(\lambda)\,\phi(-\lambda)$ equated to nothing, gives an equation for λ^2, whose signs are alternately positive and negative, which therefore, by Des Cartes's rule, cannot have a negative root. The above constitutes Sylvester's proof that the roots of the equation $\phi(\lambda) = 0$ are all real. It is evident, from what has been just said, that no root can be of the form $\beta\sqrt{(-1)}$, and in order to see that no root can be of the form $\alpha + \beta\sqrt{(-1)}$, it is only necessary to write $a - \alpha = a'$, $b - \alpha = b'$, $c - \alpha = c'$, when the case is reduced to the preceding.

LESSON IV.

MINOR AND RECIPROCAL DETERMINANTS.

27. We have seen (Art. 16) that the minors of any determinant are connected with the corresponding constituents by the relation

$$a_1A_1 + a_2A_2 + a_3A_3 + \&c. = \Delta,$$

and these minors are connected with the other constituents by the identical relations

$$b_1A_1 + b_2A_2 + b_3A_3 + \&c. = 0,$$

$$c_1A_1 + c_2A_2 + c_3A_3 + \&c. = 0, \&c.$$

For since the determinant is equal to $a_1A_1 + a_2A_2 +$ &c., and since A_1, A_2, &c. do not contain a_1, a_2, &c., therefore $b_1A_1 + b_2A_2 +$ &c. is what the determinant would become if we were to make in it $a_1 = b_1$, $a_2 = b_2$, &c.; but the determinant would then have two columns identical, and would therefore vanish (Art. 14).

28. From the above can be deduced useful identical equations connecting the products of determinants formed with the same constituents. Thus writing down the two identical equations (Art. 3)

$$a\ (bc') + b\ (ca') + c\ (ab') = 0,$$
$$a'\ (bc') + b'\ (ca') + c'\ (ab') = 0;$$

multiplying the first by d', the second by d; and, subtracting, we have

$$(ad')\ (bc') + (bd')\ (ca') + (cd')\ (ab') = 0.$$

Similarly from the three equations

$$a\ (bc'd'') - b\ (cd'a'') + c\ (da'b'') - d\ (ab'c'') = 0,$$
$$a'\ (bc'd'') - b'\ (cd'a'') + c'\ (da'b'') - d'\ (ab'c'') = 0,$$
$$a''\ (bc'd'') - b''\ (cd'a'') + c''\ (da'b'') - d''\ (ab'c'') = 0,$$

multiplying these respectively by $(e'f'')$, $(e''f)$, (ef'), and adding, we deduce the identity

$$(ae'f'')(bc'd'') - (be'f'')(cd'a'') + (ce'f'')\ (da'b'') - (de'f'')(ab'c'') = 0;$$

and so on.

29. We can now briefly write the solution of a system of linear equations

$$a_1x + b_1y + c_1z + \&c. = \xi,$$
$$a_2x + b_2y + c_2z + \&c. = \eta,$$
$$a_3x + b_3y + c_3z + \&c. = \zeta, \&c.,$$

for, multiply the first by A_1, the second by A_2, &c., and add, the coefficients of y, z, &c. will vanish identically, while the coefficient of x will be $a_1A_1 + a_2A_2 +$ &c., which is the determinant formed out of the coefficients on the left-hand side of

the equation, which we shall call Δ. Thus we get

$$\Delta x = A_1\xi + A_2\eta + A_3\zeta + \&c.,$$
$$\Delta y = B_1\xi + B_2\eta + B_3\zeta + \&c.,$$
$$\Delta z = C_1\xi + C_2\eta + C_3\zeta + \&c., \&c.$$

30. The *reciprocal* of a given determinant is the determinant whose constituents are the minors corresponding to each constituent of the given one. Thus the reciprocal of $(a_1b_2c_3)$ is

$$\begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{vmatrix},$$

where A_1, B_1, &c., have the meanings already explained. If we call this reciprocal Δ', and multiply it by the original determinant Δ, by the rule of Art. 22, we get

$$\begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1, & a_2A_1 + b_2B_1 + c_2C_1, & a_3A_1 + b_3B_1 + c_3C_1 \\ a_1A_2 + b_1B_2 + c_1C_2, & a_2A_2 + b_2B_2 + c_2C_2, & a_3A_2 + b_3B_2 + c_3C_2 \\ a_1A_3 + b_1B_3 + c_1C_3, & a_2A_3 + b_2B_3 + c_2C_3, & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}.$$

But (Art. 27) $a_1A_1 + b_1B_1 + c_1C_1 = \Delta$, $a_1A_2 + b_1B_2 + c_1C_2 = 0$, &c. This determinant, therefore, reduces to

$$\begin{vmatrix} \Delta, & 0, & 0 \\ 0, & \Delta, & 0 \\ 0, & 0, & \Delta \end{vmatrix} = \Delta^3.$$

Hence $(a_1b_2c_3)(A_1B_2C_3) = (a_1b_2c_3)^3$; therefore $(A_1B_2C_3) = (a_1b_2c_3)^2$. And in general, $\Delta'\Delta = \Delta^n$; therefore $\Delta' = \Delta^{n-1}$.

31. If we take the second system of equations in Art. 29, and solve these back again for ξ, η, &c., in terms of Δx, Δy, &c., we get

$$\Delta'\xi = a_1\Delta x + b_1\Delta y + c_1\Delta z + \&c.,$$

where a_1, b_1, c_1 are the minors of the reciprocal determinant. But these values for ξ, η, ζ, &c., must be identical with the expressions originally given; hence, remembering that $\Delta' = \Delta^{n-1}$, we get, by comparison of coefficients,

$$a_1 = \Delta^{n-2}a_1, \quad b_1 = \Delta^{n-2}b_1, \quad c_1 = \Delta^{n-2}c_1, \&c.,$$

which express, in terms of the original coefficients, the first minors of the reciprocal determinant.

32. We have seen that, considering any one column a of a determinant, every element contains as a factor a constituent from that column, and therefore the determinant can be written in the form $\Sigma a_p A_p$. In like manner, considering any two columns a, b of the determinant, it can be written in the form $\Sigma (a_p b_q) A_{p,q}$, where the sum $\Sigma (a_p b_q)$ is intended to express all possible determinants which can be formed by taking two rows of the given two columns.

For every element of the determinant contains as factors a constituent from the column a, and another from the column b; and any term $a_p b_q c_r d_s$, &c., must, by the rule of signs, be accompanied by another, $-a_q b_p c_r d_s$, &c. Hence we see that the form of the determinant is $\Sigma (a_p b_q) A_{p,q}$; and, by the same reasoning as in Art. 16, we see that the multiplier $A_{p,q}$ is the minor formed by omitting the two rows and columns in which a_p, b_q occur.

In like manner, considering any p columns of the determinant, it can be expressed as the sum of all possible determinants that can be formed by taking any p rows of the selected columns, and multiplying the minor formed with them, by the *complemental* minor; that is to say, the minor formed by erasing these rows and columns. For example,

$$\begin{aligned}(a_1b_2c_3d_4e_5) = {} & (a_1b_2)(c_3d_4e_5) - (a_1b_3)(c_2d_4e_5) + (a_1b_4)(c_2d_3e_5) - (a_1b_5)(c_3d_4e_2) \\ & + (a_2b_3)(c_1d_4e_5) - (a_2b_4)(c_1d_3e_5) + (a_2b_5)(c_1d_3e_4) + (a_3b_4)(c_1d_2e_5) \\ & - (a_3b_5)(c_1d_2e_4) + (a_4b_5)(c_1d_2e_3).\end{aligned}$$

The sign of each term in the above is determined without difficulty by the rule of signs (Art. 8).

It is evident, as in Art. 27, that if we write in the above a c for every b, the sum $\Sigma (a_1c_2)(c_3d_4e_5)$ must vanish identically, since it is what the determinant would become if the c column were equal to the b column.

33. The theorem of Art. 31 may be extended as follows: *Any minor of the order p which can be formed out of the inverse constituents A_1, B_1, &c. is equal to the complementary of the corresponding minor of the original determinant, multiplied by the $(p-1)^{\text{th}}$ power of that determinant.*

For example, in the case where the original determinant is of the fifth order,

$$(A_1B_2) = \Delta\,(c_3d_4e_5),\quad (A_1B_2C_3) = \Delta^2\,(d_4e_5),\ \&c.$$

The method in which the general theorem is proved will be sufficiently understood from the proof of this example. We have

$$\Delta x = A_1\xi + A_2\eta + A_3\zeta + A_4\omega + A_5\upsilon,$$
$$\Delta y = B_1\xi + B_2\eta + B_3\zeta + B_4\omega + B_5\upsilon.$$

Therefore

$$\Delta B_2 x - \Delta A_2 y = (A_1B_2)\,\xi + (A_3B_2)\,\zeta + (A_4B_2)\,\omega + (A_5B_2)\,\upsilon.$$

But we can get another expression for x in terms of the same five quantities y, ξ, ζ, ω, υ. For, consider the original equations,

$$\xi = a_1x + b_1y + c_1z + d_1w + e_1u,$$
$$\zeta = a_3x + b_3y + c_3z + d_3w + e_3u,$$
$$\omega = a_4x + b_4y + c_4z + d_4w + e_4u,$$
$$\upsilon = a_5x + b_5y + c_5z + d_5w + e_5u,$$

and eliminate z, w, u, we get

$$(a_1c_3d_4e_5)\,x + (b_1c_3d_4e_5)\,y = (c_3d_4e_5)\,\xi - (c_4d_5e_1)\,\zeta + (c_5d_1e_3)\,\omega - (c_1d_3e_4)\,\upsilon;$$

and since $(a_1c_3d_4e_5)$ is by definition $= B_2$, comparing these equations with the former, we find $(A_1B_2) = \Delta\,(c_3d_4e_5)$, &c., Q.E.D.

Ex. 1. If a determinant vanish, its minors A_1, A_2, &c, are respectively proportional to B_1, B_2, &c. For we have just proved that $A_1B_2 - A_2B_1 = \Delta C$, where C is the second minor obtained by suppressing the first two rows and columns. If then $\Delta = 0$, we have $A_1 : A_2 :: B_1 : B_2$, &c.

Ex. 2. A particular example of the above, which is of frequent occurrence, is obtained by applying these principles to the determinant considered, Ex. 16, p. 17. We thus find, using the notation of that example $\binom{\alpha}{\alpha}\binom{\beta}{\beta} - \binom{\alpha}{\beta}^2 = \Delta\begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix}$; (see *Surfaces*, Art. 80).

Ex. 3. As in Art. 32, the determinant $(a_1b_2c_3d_4)$ may be written

$$(a_1b_2)\,(c_3d_4) + (a_1b_3)\,(c_4d_2) + (a_1b_4)\,(c_2d_3) + (a_3b_4)\,(c_1d_2) + (a_4b_2)\,(c_1d_3) + (a_2b_3)\,(c_1d_4).$$

Ex. 4. If six arbitrary quantities p, q, r, s, t, u be assumed, and we denote by (ab) the quantity

$$p\,(a_2b_3) + q\,(a_3b_1) + r\,(a_1b_2) + s\,(a_1b_4) + t\,(a_2b_4) + u\,(a_3b_4).$$

similarly letting $(cd) = p(c_2d_3) + q(c_3d_1) + r(c_1d_2) + s(c_1d_4) + t(c_2d_4) + u(c_3d_4)$, &c., with like meanings for six such functions, then it is easily seen by the identical vanishing of groups, as in Art. 32, when the form in Ex. 3 is written with two or more suffixes the same, that

$$(bc)(ad) + (ca)(bd) + (ab)(cd) = (ps + qt + ru)(a_1b_2c_3d_4).$$

See *Surfaces*, Art. 57*e*.

Ex. 5. The homographic relation between two sets of four quantities $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ is found from the relation of a one-one correspondence between them, viz. $Aaa' + Ba + Ca' + D = 0$, &c. (*Conics*, Art. 331) in the form of the vanishing of the determinant

$$\begin{vmatrix} \alpha\alpha', & \alpha, & \alpha', & 1 \\ \beta\beta', & \beta, & \beta', & 1 \\ \gamma\gamma', & \gamma, & \gamma', & 1 \\ \delta\delta', & \delta, & \delta', & 1 \end{vmatrix}.$$

Calculating this by Ex. 3, it is found

$$= (\beta' - \gamma')(\alpha' - \delta')(\beta\gamma + \alpha\delta) + (\gamma' - \alpha')(\beta' - \delta')(\gamma\alpha + \beta\delta) + (\alpha' - \beta')(\gamma' - \delta')(\alpha\beta + \gamma\delta),$$

but this is evidently

$$\begin{vmatrix} \beta\gamma + \alpha\delta, & 1, & \beta'\gamma' + \alpha'\delta' \\ \gamma\alpha + \beta\delta, & 1, & \gamma'\alpha' + \beta'\delta' \\ \alpha\beta + \gamma\delta, & 1, & \alpha'\beta' + \gamma'\delta' \end{vmatrix};$$

and since $(\beta' - \gamma')(\alpha' - \delta') + (\gamma' - \alpha')(\beta' - \delta') + (\alpha' - \beta')(\gamma' - \delta') = 0$,

the determinant may also be written with the following abbreviations

$$A = (\beta - \gamma)(\alpha - \delta), \quad B = (\gamma - \alpha)(\beta - \delta), \quad C = (\alpha - \beta)(\gamma - \delta),$$

$$A' = (\beta' - \gamma')(\alpha' - \delta'), \quad B' = (\gamma' - \alpha')(\beta' - \delta'), \quad C' = (\alpha' - \beta')(\gamma' - \delta'),$$

in any of the forms $B'C - BC' = C'A - CA' = A'B - AB'$.

This leads to the expansion of the involution determinant (Ex. 7, p. 25) when $\delta = \alpha'$, $\delta' = \alpha$, in the form there given among others.

Ex. 6.

$$\begin{vmatrix} \sin\alpha\sin(\alpha+\theta), & \sin\alpha, & \sin(\alpha+\theta), & 1 \\ \sin\beta\sin(\beta+\theta), & \sin\beta, & \sin(\beta+\theta), & 1 \\ \sin\gamma\sin(\gamma+\theta), & \sin\gamma, & \sin(\gamma+\theta), & 1 \\ \sin\delta\sin(\delta+\theta), & \sin\delta, & \sin(\delta+\theta), & 1 \end{vmatrix} = 2\sin\theta\sin(\sigma+\theta) \begin{vmatrix} \cos^2\sigma_1, & \cos\sigma_1, & 1 \\ \cos^2\sigma_2, & \cos\sigma_2, & 1 \\ \cos^2\sigma_3, & \cos\sigma_3, & 1 \end{vmatrix}$$

$$= 16\sin\theta\sin(\sigma+\theta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\alpha-\delta)\sin\tfrac{1}{2}(\beta-\delta)\sin\tfrac{1}{2}(\gamma-\delta),$$

as is easily seen by applying Ex. 3, coupling first and fourth columns and second and third, and introducing the abbreviations

$$2\sigma = \alpha+\beta+\gamma+\delta, \quad 2\sigma_1 = \alpha+\beta-\gamma-\delta, \quad 2\sigma_2 = \alpha-\beta+\gamma-\delta, \quad 2\sigma_3 = \alpha-\beta-\gamma+\delta.$$

Ex. 7. The theorem of Art. 33 may be established otherwise as follows. Let

$$\begin{vmatrix} a_1, & a_2, & \ldots a_n \\ b_1, & b_2, & \ldots b_n \\ \ldots\ldots & & \\ l_1, & l_2, & \ldots l_n \end{vmatrix} = \Delta, \quad \text{then} \quad \begin{vmatrix} A_1, & A_2, & \ldots A_n \\ B_1, & B_2, & \ldots B_n \\ \ldots\ldots & & \\ L_1, & L_2, & \ldots L_n \end{vmatrix} = \Delta^{n-1}$$

is by Art. 30 the determinant of the reciprocal, or, as Cauchy called it, the adjoint system.

Now we have the following forms for the product

$$\begin{vmatrix} B_2, \ldots B_n \\ C_2, \ldots C_n \\ \ldots\ldots\ldots \\ L_2, \ldots L_n \end{vmatrix} \Delta = \begin{vmatrix} \Delta, & A_2, & A_3, \ldots A_n \\ 0, & B_2, & B_3, \ldots B_n \\ & \ldots\ldots\ldots\ldots\ldots \\ 0, & L_2, & L_3, \ldots L_n \end{vmatrix} = \begin{vmatrix} a_1A_1, & A_2, \ldots A_n \\ a_1B_1, & \ldots\ldots\ldots B_n \\ \ldots\ldots\ldots\ldots\ldots \\ a_1L_1, & L_2, \ldots L_n \end{vmatrix} = a_1\Delta^{n-1};$$

whence a factor Δ can be removed. Similarly the product

$$\begin{vmatrix} C_3, \ldots C_n \\ D_3, \ldots D_n \\ \ldots\ldots\ldots \\ L_3, \ldots L_n \end{vmatrix} \Delta^2 = \begin{vmatrix} \Delta, & 0, & A_3, & A_4, \ldots A_n \\ 0, & \Delta, & B_3, & B_4, \ldots B_n \\ 0, & 0, & C_3, & C_4, \ldots C_n \\ & & \ldots\ldots\ldots\ldots\ldots \\ 0, & 0, & L_3, & L_4, \ldots L_n \end{vmatrix} = \begin{vmatrix} a_1A_1 + a_2A_2, & b_1A_1 + b_2A_2, & A_3, & A_n \\ a_1B_1 + a_2B_2, & b_1B_1 + b_2B_2, & B_3, & B_n \\ a_1C_1 + a_2C_2, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1L_1 + a_2L_2, & b_1L_1 + b_2L_2, & L_3, & L_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \Delta^{n-1};$$

from which the common factor Δ^2 can be removed. And in like manner generally for a minor of order p.

LESSON V.

SYMMETRICAL AND SKEW SYMMETRICAL DETERMINANTS.

34. In this lesson it is convenient to employ the double suffix notation, and to write the constituents a_{11}, a_{12}, &c.; and we, therefore, begin by expressing in this notation some of the results already obtained. We denote the constituents of the reciprocal determinant by α_{11}, α_{12}, &c., where, if a_{rs} be any constituent of the original, α_{rs} is the minor obtained by erasing the row and column which contain that constituent. The equations of Art. 27 may then be written

$$a_{r_1}\alpha_{r_1} + a_{r_2}\alpha_{r_2} + a_{r_3}\alpha_{r_3} + \&c. = \Delta,$$

$$a_{r_1}\alpha_{r'_1} + a_{r_2}\alpha_{r'_2} + a_{r_3}\alpha_{r'_3} + \&c. = 0,$$

or more briefly $\Sigma_s a_{r_s}\alpha_{r_s} = \Delta$, $\Sigma_s a_{r_s}\alpha_{r'_s} = 0$; that is to say, the sum of the products $a_{r_s}\alpha_{r'_s}$ (where we give every value to s from 1 to n) is $= 0$, when r and r' are different, and $= \Delta$ when $r = r'$.

Since any constituent a_{r_s} enters into the determinant only in the first degree, it is obvious that the factor α_{r_s}, which multiplies it, is the differential coefficient of the determinant taken with respect to a_{r_s}; similarly, that the second minor (Art. 32), which multiplies the product of two constituents a_{mn}, a_{r_s}, is the second differential coefficient of the determinant taken with respect to these two constituents, &c.

If any of the constituents be functions of any variable x, the entire differential of the determinant, with regard to that variable, is evidently $\alpha_{11}\frac{da_{11}}{dx} + \alpha_{12}\frac{da_{12}}{dx} + \&c.$, and may be written down as the sum of n determinants in each of which one row (or column) only, differs from that in the original determinant by having the constituents of the latter differentiated.

Ex. If u_1, v_1, &c. denote the first differentials of u, v, &c. with respect to x; u_2, v_2 the second differentials, &c., prove

$$\frac{d}{dx}\begin{vmatrix} u, & v, & w \\ u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \end{vmatrix} = \begin{vmatrix} u, & v, & w \\ u_1, & v_1, & w_1 \\ u_3, & v_3, & w_3 \end{vmatrix}.$$

The differential is the sum of the nine products of the differential of each term by the minor obtained by suppressing that term,

$$(a_{11}u_1 + a_{12}v_1 + a_{13}w_1) + (a_{21}u_2 + a_{22}v_2 + a_{23}w_2) + (a_{31}u_3 + a_{32}v_3 + a_{33}w_3).$$

But the first three terms denote the result of changing in the given determinant the first row into u_1, v_1, w_1, and therefore vanish; the second three terms vanish as denoting the result of changing the second row into u_2, v_2, w_2; and there only remain the last three terms which denote the result of changing the last row into u_3, v_3, w_3. The same proof evidently applies to the similar determinant of the n^{th} order formed with n functions.

35. The determinant is said to be symmetrical (Art. 10) when every two conjugate constituents are equal ($\alpha_{rs} = \alpha_{sr}$). In this case it is to be observed, that the corresponding minors will also be equal ($\alpha_{rs} = \alpha_{sr}$); for it easily appears that the determinant got by suppressing the r^{th} row and the s^{th} column differs only by an interchange of rows for columns from that got by suppressing the s^{th} row and the r^{th} column. It appears from the last article, that if any constituent a_{sr} were given as any function of its conjugate a_{rs}, the differential coefficient of the determinant, with regard to a_{rs}, would be $\alpha_{rs} + \alpha_{sr}\frac{da_{sr}}{da_{rs}}$. In the present case then, where $a_{rs} = a_{sr}$, $\alpha_{rs} = \alpha_{sr}$, the differential coefficient of the determinant, with regard to a_{rs}, is $2\alpha_{rs}$. The differential coefficient, however, with respect to one of the constituents in the leading diagonal a_{rr} remains as before α_{rr}, since such a term has no conjugate distinct from itself.

36. If, as before, α_{rs} denote the first minor of any determinant answering to any constituent a_{rs}, and if β'_{tu} denote the first minor of the determinant α_{rs} answering to any constituent

a_{ik}, which will, of course, be a second minor of the original determinant, then this last may be written

$$\Delta = a_{rs}\alpha_{rs} - \Sigma_{ik} a_{rk} a_{is} \beta_{ik};$$

where we are to give i every value except r, and k every value except s. For, any element of the determinant which does not contain the constituent a_{rs} must contain some other constituent from the r^{th} row, and some other from the s^{th} column; that is to say, must contain a product such as $a_{rk}a_{is}$ where i and k are two numbers different from r and s respectively. But as we have already seen, the aggregate of all the terms which multiply a_{rs} is α_{rs}; and the coefficient of $a_{rk}a_{is}$ (by Art. 32) differs only in sign from that of $a_{rs}a_{ik}$; that is to say, differs only in sign from the coefficient of a_{ik} in α_{rs}. Therefore $-\beta_{ik}$ is the value of the coefficient in question.

Thus then if we have calculated a symmetric determinant of the $n-1^{th}$ order, we can see what additional terms occur in the determinant of the n^{th} order. Let Δ be the determinant, D that obtained by suppressing the outside row and column, β_{rs} any minor of the latter, and we have

$$\Delta = D a_{nn} - \Sigma_r \alpha^2_{nr} \beta_{rr} - 2\Sigma_{rs} a_{nr} a_{ns} \beta_{rs},$$

where r is supposed to be different from s, and every value is to be given to r and s from 1 to $n-1$.

Again, we have occasion often, as at p. 17, to deal with determinants such as

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \lambda_1 \\ a_{12}, & a_{22}, & a_{23}, & \lambda_2 \\ a_{13}, & a_{23}, & a_{33}, & \lambda_3 \\ \lambda_1, & \lambda_2, & \lambda_3 & \end{vmatrix},$$

obtained by bordering a symmetric determinant horizontally and vertically with the same constituents. This is in fact a symmetric determinant of the order one higher, the last term vanishing, and is

$$-\{\alpha_{11}\lambda_1^2 + \alpha_{22}\lambda_2^2 + \alpha_{33}\lambda_3^2 + 2\alpha_{23}\lambda_2\lambda_3 + 2\alpha_{13}\lambda_1\lambda_3 + 2\alpha_{12}\lambda_1\lambda_2\},$$

or generally $\quad -\Sigma_r \alpha_{rr}\lambda_r^2 - 2\Sigma_{rs}\alpha_{rs}\lambda_r\lambda_s.$

37. *If any symmetric determinant vanishes, the same determinant bordered as in the last article is, with sign changed if*

need be, a perfect square, when considered as a function of λ_1, λ_2, λ_3, *&c.* We saw (Art. 33, Ex. 1) that when the determinant vanishes $\alpha_{11}\alpha_{22} = \alpha^2_{12}$, &c., whence it is evident that $\alpha_{11}\alpha_{22}$, &c. must have all the same sign, and we have generally $\alpha_{rs} = \pm\sqrt{(\alpha_{rr}\alpha_{ss})}$. Further, since it was shewn in the same example that when a determinant vanishes, the constituents of the reciprocal determinant in the second row are proportional to those in the first, it follows that the signs to be given to the radicals are not all arbitrary. If, for instance, in the above we write $\alpha_{12} = +\sqrt{(\alpha_{11}\alpha_{22})}$, $\alpha_{13} = +\sqrt{(\alpha_{11}\alpha_{33})}$, then we are forced to give the positive sign also to the square root in $\alpha_{23} = \sqrt{(\alpha_{22}\alpha_{33})}$. Substituting, then, these values in the result of the last article, it becomes, if α_{11}, &c. be positive, the negative square,

$$-\{\lambda_1\sqrt{(\alpha_{11})} + \lambda_2\sqrt{(\alpha_{22})} + \lambda_3\sqrt{(\alpha_{33})} + \&c.\}^2,$$

and if α_{11}, &c. be negative the determinant is a positive square.

What has been just proved may be stated a little differently. We may consider the bordered determinant as the original determinant; of which, that obtained by suppressing the row and column containing λ is a first minor, and α_{33} obtained by suppressing the next outside row and column is a second minor. And what we have proved with respect to any symmetrical determinant wanting the last term a_{nn}, is, that if the first minor obtained by erasing the outside row and column vanish, then the determinant itself and the second minor, similarly obtained, must have opposite signs. And this will be equally true if a_{nn} does not vanish. For in the expansion of the determinant, a_{nn} is multiplied by the first minor, which vanishes by hypothesis, and therefore the presence or absence of a_{nn} does not affect the truth of the result.

38. A *skew symmetric* determinant is one in which each constituent is equal to its conjugate with its sign changed. Constituents a_{rr} in the leading diagonal, being each its own conjugate, must in this case vanish; otherwise each could not be equal to itself with sign changed.

A skew symmetrical determinant of odd degree vanishes. For if we multiply each row by -1; in other words, if we change

the sign of every term, it is easy to see that we get the same result as if we were to read the columns of the original determinant as rows, and *vice versâ*. Thus, then, a skew symmetrical determinant is not altered when multiplied by $(-1)^n$; and, therefore, when n is odd, such a determinant must vanish.

It is easy to see that the minor α_{sr} differs by the sign of each constituent from the minor α_{rs}, and therefore $\alpha_{sr} = (-1)^{n-1}\alpha_{rs}$. Hence $\alpha_{sr} = \alpha_{rs}$ when n is odd, and is equal with contrary sign when n is even. α_{rr} is itself a skew symmetric determinant, and therefore vanishes when the original determinant is of even degree.

The differential coefficient of the determinant, with regard to any constituent a_{rs}, being $\alpha_{rs} + \alpha_{sr}\dfrac{da_{sr}}{da_{rs}}$ is $\alpha_{rs} - \alpha_{sr}$. When therefore n is even it is $= 2\alpha_{rs}$ and when n is odd it vanishes.

39. *Every skew symmetrical determinant of even degree is a perfect square.* (Prof. Cayley).

It was shewn (Art. 36) that any determinant is

$$= a_{nn}\alpha_{nn} - \Sigma_{rs}a_{ns}a_{rn}\beta_{rs},$$

and in the present case a_{nn} vanishes, as does also α_{nn}, which is a skew symmetric of odd degree. On this account therefore we have, as in Art. 37, $\beta^2_{rs} = \beta_{rr}\beta_{ss}$; and therefore, exactly as in that article, the determinant is shown to be

$$\{a_{1n}\sqrt{(\beta_{11})} + a_{2n}\sqrt{(\beta_{22})} + a_{3n}\sqrt{(\beta_{33})} + \&c.\}^2.$$

The determinant is therefore a perfect square if β_{11}, β_{22} are perfect squares, but these are skew symmetrics of the order $n-2$. Hence the theorem of this article is true for determinants of order n if true for those of order $n-2$, and so on. But it is evidently true for the determinant of the second order $\begin{vmatrix} 0, & a_{12} \\ -a_{12}, & 0 \end{vmatrix}$, which is $= a^2_{12}$. Hence it is generally true.

40. We have seen that the square root of the determinant contains one term $a_{n-1,n}\sqrt{(\beta_{n-1,n-1})}$, where $\beta_{n-1,n-1}$ contains no constituent with either of the suffixes $n-1$ or n. But taking any two of the remaining suffixes, such as $n-3$, $n-2$, we see that

$\sqrt{(\beta_{n-1,n-1})}$ contains a term $a_{n-2,n-2}\sqrt{(\gamma_{n-3,n-3})}$, where $\gamma_{n-3,n-3}$ contains none of the four suffixes, of which account has already been taken. Proceeding in this manner we see that the square root will be the sum of a number of terms such as $a_{12}a_{34}a_{56}\ldots a_{n-1,n}$; each of which is the product of $\frac{1}{2}n$ constituents, and in which no suffix is repeated. The form, however, obtained in the last article $a_{1n}\sqrt{(\beta_{11})} \pm a_{2n}\sqrt{(\beta_{22})}$ + &c. does not show what sign is to be affixed to each term. Thus if the method of the last article be applied to the skew symmetric of the fourth order, its square root appears to be $a_{12}a_{34} \pm a_{13}a_{24} \pm a_{14}a_{23}$; but it has not been shown which signs we are to choose. This, however, will appear from the following considerations: When in the given determinant we interchange any two suffixes 1, 2, since this amounts to a transposition of the first and second row, and also of the first and second column, the determinant is not altered. Its square root then must be a function, such that if we interchange any two suffixes it will remain unaltered, or at most change sign. But that it *will* change sign is evident on considering any term $a_{12}a_{34}$, &c. which, if we interchange the suffixes 1 and 2, becomes $a_{21}a_{34}$, &c.; that is to say, changes sign, since $a_{21} = -a_{12}$. It follows then, in the particular example just considered, that the signs of the terms are $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$; for if we give the second term a positive sign, the interchange of 2 and 3, which alters the sign of the last term, would leave the first two unchanged. And generally the rule is, that the square root is the sum of all possible terms derived from $a_{12}a_{34}\ldots a_{n-1,n}$ by interchange of the suffixes 2, 3, ...n, where, as in determinants, we change sign with every permutation. But it is possible, and the better course is, to effect the interchanges in such manner that the signs shall be each of them +; thus, in the particular example, the expression may be written $a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}$.

Ex. 1. To write out the skew symmetrical determinant of the sixth order. Denoting by (1234) the expression just given with all signs positive for the square root of that of the fourth order, if we proceed always in the same cyclical order we write down the expressions (3456), (4562), (5623), (6234), (2345), and the expression required is the square of the sum of the fifteen terms contained in

$$a_{12}(3456) + a_{13}(4562) + a_{14}(5623) + a_{15}(6234) + a_{16}(2345).$$

Ex. 2. If the constituents of a symmetrical determinant be each increased by k times the corresponding constituents of a skew symmetrical, the determinant so formed when expanded contains only even powers of k. For, altering k to $-k$ does not change its value.

For instance (see *Surfaces*, Art. 80*c*) we may write

$$\begin{vmatrix} a, & h-kz, & g+ky, & l-k\alpha \\ h+kz, & b, & f-kx, & m-k\beta \\ g-ky, & f+kx, & c, & n-k\gamma \\ l+k\alpha, & m+k\beta, & n+k\gamma, & d, \end{vmatrix} = \Delta + k^2\Psi + k^4(\alpha x+\beta y+\gamma z)^2.$$

The same proof shews that if conjugate constituents of a determinant be imaginary quantities, the value of the determinant is real.

Ex. 3. The square of any determinant of even order may be written as a skew symmetrical determinant. For instance, calculate the product

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} \begin{vmatrix} a_4, & -a_3, & a_2, & -a_1 \\ b_4, & -b_3, & b_2, & -b_1 \\ c_4, & -c_3, & c_2, & -c_1 \\ d_4, & -d_3, & d_2, & -d_1 \end{vmatrix},$$

and it is found to be of the skew symmetrical form. Or, again, for any multipliers p, q, r, s, t, u, form the determinant product of

$$\begin{vmatrix} 0, & -r, & q, & -s \\ r, & 0, & -p, & -t \\ -q, & p, & 0, & -u \\ s, & t, & u, & 0 \end{vmatrix} \begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix},$$

and multiply it by

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3 & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix}.$$

the product may be written

$$\begin{vmatrix} 0\ , & (ab), & (ac), & (ad) \\ (ba), & 0\ , & (bc), & (bd) \\ (ca), & (cb), & 0\ , & (cd) \\ (da), & (db), & (dc), & 0 \end{vmatrix}$$

in the notation of Ex. 4, Art. 33. Since $(ab) = -(ba)$ &c., we have thus proved

$$\{(bc)(ad) + (ca)(bd) + (ab)(cd)\}^2 = (ps + qt + ru)^2 (a_1b_2c_3d_4)^2,$$

which was otherwise seen in that Example.

41. We can reduce to the calculation of skew symmetric determinants the calculation of what Prof. Cayley calls a *skew* determinant, in which, though the conjugate terms are equal with opposite signs, $a_{ik} = -a_{ki}$, yet the leading terms a_{ii}, a_{kk}, &c. do not vanish. We shall suppose, for simplicity, that these leading terms all have a common value λ. We prefix the following lemma: If in any determinant we denote by D the result of making all the leading terms $= 0$, by D_i what the minor corresponding to a_{ii} becomes when the leading terms are all made $= 0$, by D_{ik} what the second minor corresponding to $a_{ii}a_{kk}$ becomes when the leading terms vanish, &c., then the

given determinant, expanded as far as the leading constituents are concerned, is

$$\Delta = D + \Sigma a_{ii} D_i + \Sigma a_{ii} a_{kk} D_{ik} + \ldots + a_{11} a_{22} \ldots a_{nn},$$

where, in the first sum i has any value from 1 to n, in the second sum i, k are any binary combinations of these numbers, &c.

For, the part of the determinant which contains no leading constituent is evidently D; the terms which contain a_{ii} are $a_{ii} A_{ii}$, where A_{ii} is the corresponding minor, hence the terms which contain a_{ii} and no other leading constituents are got by making the leading constituents $= 0$ in A_{ii}; and so for the other terms.

42. If this lemma be applied to the case of the skew determinant defined in the last article, all the terms D_i, D_{ik}, &c. are skew symmetric determinants; of which, those of odd order vanish, while those of even order are perfect squares. The term $a_{11} a_{22} \ldots a_{nn}$ is λ^n, and the determinant is

$$\Delta = \lambda^n + \lambda^{n-2} \Sigma D_2 + \lambda^{n-4} \Sigma D_4 + \&c.,$$

where D_2, D_4, &c. denote skew symmetrical determinants of the second, fourth, &c. orders formed from the original in the manner explained in the last article.

Ex. 1. $\begin{vmatrix} \lambda, & a_{12}, & a_{13} \\ a_{21}, & \lambda, & a_{23} \\ a_{31}, & a_{32}, & \lambda \end{vmatrix}$, where $a_{21} = -a_{12}$, &c.

$$\text{is} = \lambda^3 + \lambda (a_{12}^2 + a_{13}^2 + a_{23}^2).$$

Ex. 2. The similar skew determinant of the fourth order expanded is

$$\lambda^4 + \lambda^2 (a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2) + (a_{12} a_{34} + a_{13} a_{42} + a_{14} a_{23})^2.$$

43. Prof. Cayley (*Crelle*, vol. XXXII., p. 119) has applied the theory of skew determinants to that of orthogonal substitutions, of which we shall here give some account. It is known (see *Surfaces*, p. 10) that when we transform from one set of three rectangular axes to another, if a, b, c, &c. be the direction-cosines of the new axes, and if

$$X = ax + by + cz, \quad Y = a'x + b'y + c'z, \quad Z = a''x + b''y + c''z;$$

that we have $X^2 + Y^2 + Z^2 = x^2 + y^2 + z^2$,

whence $a^2 + a'^2 + a''^2 = 1$, &c., $ab + a'b' + a''b'' = 0$, &c.;

that also we have

$$x = aX + a'Y + a''Z, \quad y = bX + b'Y + b''Z, \quad z = cX + c'Y + c''Z,$$

and that we have the determinant formed by

$$a, b, c;\ a', b', c';\ \&c. = \pm 1.$$

It is also useful (in studying the theory of rotation for example) instead of using nine quantities a, b, c, &c. connected by six relations, to express all in terms of three independent variables. Now all this may be generalized as follows: If we have a function of any number of variables, it can be transformed by a linear substitution by writing

$$x = a_{11}X + a_{12}Y + a_{13}Z + \&c.,\ y = a_{21}X + a_{22}Y + a_{23}Z + \&c., \&c.,$$

and the substitution is called orthogonal if we have

$$x^2 + y^2 + z^2 + \&c. = X^2 + Y^2 + Z^2 + \&c.,$$

which implies the equations

$$a_{11}^2 + a_{21}^2 + \&c. = 1,\quad a_{11}a_{12} + a_{21}a_{22} + \&c. = 0, \&c.$$

Thus the n^2 quantities a_{11}, &c. are connected by $\frac{1}{2}n(n+1)$ relations, and there are only $\frac{1}{2}n(n-1)$ of them independent.

We have then conversely

$$X = a_{11}x + a_{21}y + a_{31}z + \&c.,\quad Y = a_{12}x + a_{22}y + a_{32}z + \&c., \&c.,$$

equations which are immediately verified by substituting on the right-hand side of the equations for x, y, z, &c. their values. And hence, the equation $X^2 + Y^2 + \&c. = x^2 + y^2 + \&c.$ gives us the new system of relations

$$a_{11}^2 + a_{12}^2 + \&c. = 1,\quad a_{11}a_{21} + a_{12}a_{22} + \&c. = 0.$$

Lastly, forming by the ordinary rule for multiplication of determinants, the square of the determinant formed with the n^2 quantities a_{11}, &c., each constituent of the square vanishes except the leading constituents, which are each $= 1$. The value of the square is therefore $= 1$. Thus the theorems which we know to be true in the case of determinants of the third order are generally true, and it only remains to shew how to express the n^2 quantities in terms of $\frac{1}{2}n(n-1)$ independent quantities. This we shall effect by a method employed by M. Hermite for the more general problem of the transformation of a quadric function into itself. See his paper 'Remarques, &c.,' *Camb. and Dub. Math. Jour.*, vol. IX. (1854), p. 63.

44. Let us suppose that we have a skew determinant of the $(n-1)^{\text{th}}$ order, b_{11}, b_{12}, &c. where $b_{ik} = -b_{ki}$ and $b_{11} = b_{22} = b_{ii} = 1$; and let us suppose that we form with these constituents the two different sets of linear substitutions,

$$x = b_{11}\xi + b_{12}\eta + b_{13}\zeta + \&c., \quad X = b_{11}\xi + b_{21}\eta + b_{31}\zeta + \&c.,$$
$$y = b_{21}\xi + b_{22}\eta + b_{23}\zeta + \&c., \quad Y = b_{12}\xi + b_{22}\eta + b_{32}\zeta + \&c.,$$
$$z = b_{21}\xi + b_{32}\eta + b_{33}\zeta + \&c., \quad Z = b_{13}\xi + b_{23}\eta + b_{33}\zeta + \&c.,$$

from adding which equations we have, in virtue of the given relations between b_{11}, b_{12}, &c.,

$$x + X = 2\xi, \quad y + Y = 2\eta, \text{ \&c.}$$

If now the first set of equations be solved for ξ, η, &c. in terms of x, y, &c., we find, by Art. 29,

$$\Delta\xi = \beta_{11}x + \beta_{21}y + \beta_{31}z + \&c., \quad \Delta\eta = \beta_{12}x + \beta_{22}y + \&c.$$

(where β_{11}, β_{12}, &c. are minors of the determinant in question); and putting for 2ξ, $x + X$, &c., these equations give

$$\Delta X = (2\beta_{11} - \Delta)x + 2\beta_{21}y + 2\beta_{31}z + \&c.,$$
$$\Delta Y = 2\beta_{12}x + (2\beta_{22} - \Delta)y + 2\beta_{32}z + \&c., \&c.,$$

which express X, Y, &c. in terms of x, y, &c. But if we had solved from the second set of equations for ξ, η, &c. in terms of X and Y, &c. we should have found

$$\Delta\xi = \beta_{11}X + \beta_{12}Y + \beta_{13}Z + \&c., \quad \Delta\eta = \beta_{21}X + \beta_{22}Y + \beta_{23}Z + \&c.,$$

whence, as before,

$$\Delta x = (2\beta_{11} - \Delta)X + 2\beta_{12}Y + 2\beta_{13}Z + \&c.,$$
$$\Delta y = 2\beta_{21}X + (2\beta_{22} - \Delta)Y + 2\beta_{23}Z + \&c.$$

Thus, then, if we write

$$\frac{2\beta_{11} - \Delta}{\Delta} = a_{11}, \quad \frac{2\beta_{ii} - \Delta}{\Delta} = a_{ii}; \quad \frac{2\beta_{12}}{\Delta} = a_{12}, \quad \frac{2\beta_{ik}}{\Delta} = a_{ik},$$

we have x, y, &c. connected with X, Y, &c. by the relations

$$x = a_{11}X + a_{21}Y + \&c., \quad y = a_{21}X + a_{22}Y + \&c., \&c.,$$
$$X = a_{11}x + a_{21}y + \&c., \quad Y = a_{12}x + a_{22}y + \&c., \&c.$$

We have then x, y, &c., X, Y, &c. connected by an orthogonal substitution, for if we substitute in the value of x the values

of X, Y, &c. given by the second set of equations, in order that our results may be consistent, we must have

$$a_{11}^2 + a_{12}^2 + a_{13}^2 + \&c. = 1, \quad a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} + \&c. = 0, \ \&c.$$

Thus then we have seen that taking arbitrarily the $\frac{1}{2}n(n-1)$ quantities, b_{12}, b_{13}, &c., we are able to express in terms of these the coefficients of a general orthogonal transformation of the n^{th} order.

Ex. 1. To form an orthogonal transformation of the second order. Write

$$\Delta = \begin{vmatrix} 1, & \lambda \\ -\lambda, & 1 \end{vmatrix} = 1 + \lambda^2,$$

then $\beta_{11} = \beta_{22} = 1$, $\beta_{12} = \lambda$, $\beta_{21} = -\lambda$, and our transformation is

$$(1+\lambda^2)\,x = (1-\lambda^2)\,X + 2\lambda Y, \quad (1+\lambda^2)\,X = (1-\lambda^2)\,x - 2\lambda y,$$
$$(1+\lambda^2)\,y = -2\lambda X + (1-\lambda^2)\,Y, \quad (1+\lambda^2)\,Y = 2\lambda x + (1-\lambda^2)\,y.$$

Ex. 2. To form an orthogonal transformation of the third order. Write

$$\Delta = \begin{vmatrix} 1, & \nu, & -\mu \\ -\nu, & 1, & \lambda \\ \mu, & -\lambda, & 1 \end{vmatrix} = 1 + \lambda^2 + \mu^2 + \nu^2.$$

Then the constituents of the reciprocal system are

$$\begin{vmatrix} 1+\lambda^2, & \nu+\lambda\mu, & -\mu+\lambda\nu \\ -\nu+\lambda\mu, & 1+\mu^2, & \lambda+\mu\nu \\ \mu+\lambda\nu, & -\lambda+\mu\nu, & 1+\nu^2 \end{vmatrix},$$

consequently the coefficients of the orthogonal substitution hence derived are

$$\begin{matrix} 1+\lambda^2-\mu^2-\nu^2, & 2(\nu+\lambda\mu), & 2(\lambda\nu-\mu), \\ 2(\lambda\mu-\nu), & 1+\mu^2-\lambda^2-\nu^2, & 2(\mu\nu+\lambda), \\ 2(\lambda\nu+\mu), & 2(\mu\nu-\lambda), & 1+\nu^2-\lambda^2-\mu^2, \end{matrix}$$

where each term is to be divided by $1 + \lambda^2 + \mu^2 + \nu^2$.*

45. It is easy to see that for a symmetrical determinant of the orders 1, 2, 3, 4 the number of distinct terms is $= 1, 2, 5, 17$ respectively, and the question thus arises what is the number of distinct terms in a symmetrical determinant of the order n. This number has been calculated as follows by Professor Cayley:

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* The geometric meaning of these coefficients may be stated as follows: Write $\lambda = a \tan \frac{1}{2}\theta$, $\mu = b \tan \frac{1}{2}\theta$, $\nu = c \tan \frac{1}{2}\theta$, then the new axes may be derived from the old by rotating the system through an angle θ round an axis whose direction-cosines are a, b, c. The theory of orthogonal substitutions was first investigated by Euler (*Nov. Comm. Petrop.*, vol. XV., p. 75, and vol. XX., p. 217), who gave formulæ for the transformation as far as the fourth order. The quantities λ, μ, ν, in the case of the third order, were introduced by Rodrigues, *Liouville*, vol. V., p. 405. The general theory, explained above, connecting linear transformations with skew determinants, was given by Cayley, *Crelle*, vol. XXXII., p. 119.

Consider a partially symmetrical determinant represented (see Art. 11) by the notation

$$\left|\begin{array}{c} aa \\ bb \\ cc \\ :: \\ pp' \\ qq' \end{array}\right|,$$

where in general $fg = gf$, but all the letters $p, q, \ldots$ are distinct from all the letters $p', q', \ldots$ so that these letters give rise to no equalities of conjugate terms; say, if in the bicolumn there are m rows $aa, bb, \ldots$ and n rows $pp', qq', \ldots$ this is a determinant (m, n); and in the case $n = 0$, a symmetrical determinant. And let $\phi(m, n)$ be the number of distinct terms in a determinant (m, n).

Consider *first* a determinant for which n is not 0, for instance

$$\left|\begin{array}{c} aa \\ bb \\ pp' \\ qq' \end{array}\right| , = \left|\begin{array}{cccc} aa, & ab, & ap', & aq' \\ ba, & bb, & bp', & bq' \\ pa, & pb, & pp', & pq' \\ qa, & qb, & qp', & qq' \end{array}\right|,$$

then qa, qb, qp', qq' are distinct from each other and from every other constituent of the determinant, and the whole determinant is (disregarding signs) the sum of these each multiplied into a minor determinant; the minors which multiply qa, qb are each of the form $(1, 2)$; those which multiply qp', qq' are each of the form $(2, 1)$; and we thus obtain

$$\phi(2, 2) = 2\phi(1, 2) + 2\phi(2, 1),$$

and so in general

$$\phi(m, n) = m\phi(m-1,\ n) + n\phi(m, n-1),$$

whence in particular

$$\phi(m, 1) = m\phi(m-1,\ 1) + \phi(m, 0).$$

Next, if $n = 0$, let us take for instance the symmetrical determinant

$$\left|\begin{array}{c} aa \\ bb \\ cc \\ dd \end{array}\right| , = \left|\begin{array}{cccc} aa, & ab, & ac, & ad \\ ba, & bb, & bc, & bd \\ ca, & cb, & cc, & cd \\ da, & db, & dc, & dd \end{array}\right|.$$

We have here terms multiplied by dd; $ad.da$, $bd.db$, $cd.dc$; and by the pairs of equal terms $ad.db + bd.da$, $ad.dc + cd.da$, $bd.dc + cd.db$, the other factors being in the three cases minors of the forms (3, 0), (2, 0), and (1, 1) respectively; thus we have

$$\phi(4, 0) = \phi(3, 0) + 3\phi(2, 0) + 3\phi(1, 1),$$

and so in general

$$\phi(m, 0) = \phi(m-1, 0) + (m-1)\,\phi(m-2, 0) + \tfrac{1}{2}(m-1)(m-2)\,\phi(m-3, 1),$$

which last equation combined with the foregoing

$$\phi(m, 1) = m\phi(m-1, 1) + \phi(m, 0)$$

gives the means of calculating $\phi(m, 0)$, $\phi(m, 1)$; and then the general equation $\phi(m, n) = m\phi(m-1, n) + n\phi(m, n-1)$ gives the remaining quantities $\phi(m, n)$.

It is easy to derive the equation

$$\begin{aligned} 2\phi(m, 0) - \phi(m-1, 0) - (m-1)\,\phi(m-2, 0) = {} & \phi(m-1, 0) \\ & + (m-1)\,\phi(m-2, 0) \\ & + (m-1)(m-2)\,\phi(m-3, 0) \\ & + \ldots\ldots\ldots\ldots \\ & + (m-1)\ldots 3.2.1\;\phi(0, 0). \end{aligned}$$

And hence, using the method of generating functions, and assuming

$$u = \phi(0, 0) + \frac{x}{1}\phi(1, 0) + \frac{x^2}{1.2}\phi(2, 0)\ldots + \frac{x^m}{1.2\ldots m}\phi(m, 0) + \ldots,$$

we find at once $$2\frac{du}{dx} - u - xu = \frac{u}{1-x},$$

that is $$2\frac{du}{u} = \left(1 + x + \frac{1}{1-x}\right)dx,$$

or, integrating and determining the constant, so that for $x = 0$ u shall become $= 1$, we have

$$u = \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{(1-x)}},$$

whence $\phi(m, 0)$, the number of terms in a symmetrical determinant of the order m, is

$$= 1.2\ldots m \text{ coefficient of } x^m \text{ in } \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{(1-x)}}.$$

The numerical calculation by this formula is, however, somewhat complicated; and it is practically easier to use the equations of differences directly. We thus obtain not only the values of $\phi(m, 0)$, but the series of values

$$\phi(0, 0),$$
$$\phi(1, 0),\ \phi(0, 1),$$
$$\phi(2, 0),\ \phi(1, 1),\ \phi(0, 2),$$
$$\phi(3, 0), \&c.,$$

which are found to be

1,
1, 1,
2, 2, 2,
5, 6, 6, 6,
17, 23, 24, 24, 24,
73, 109, 118, 120, 120, 120,
388, 618, 690, 714, 720, 720, 720,
&c., &c.,

as is easily verified. But as regards $\phi(m, 0)$ we may, by writing the equation in u under the slightly different form

$$2(1-x)\frac{du}{dx} - (2-x^2)u = 0,$$

obtain from it the new equation of differences

$$\phi(m, 0) = m\phi(m-1, 0) - \tfrac{1}{2}(m-1)(m-2)\phi(m-3, 0),$$

and the process of calculation is then very easy. Starting from the values $\phi(1, 0) = 1$, $\phi(2, 0) = 2$, which imply $\phi(0, 0) = 1$, we have

$$\begin{aligned}
m = 1, &\quad 1 = 1.\ 1 \\
= 2, &\quad 2 = 2.\ 1 \\
= 3, &\quad 5 = 3.\ 2 - 1.1, \\
= 4, &\quad 17 = 4.\ 5 - 3.1, \\
= 5, &\quad 73 = 5.17 - 6.2, \\
= 6, &\quad 388 = 6.73 - 10.5, \\
\&c., &\quad \&c.
\end{aligned}$$

LESSON VI.

DISCRIMINATING SYMMETRICAL DETERMINANTS.

46. If we add the quantity λ to each of the leading terms of a symmetrical determinant, and equate the result to 0, we have an equation of considerable importance in analysis.* We have already given one proof (Sylvester's) that the roots of this equation are all real (Ex. 14, p. 28), and we purpose in this Lesson to give another proof by Borchardt (see *Liouville*, vol. XII. p. 50), chiefly because the principles involved in this proof are worth knowing for their own sake. First, however, we may remark that a simple proof may be obtained by the application of a principle proved in Art. 37. Take the determinant

$$\begin{vmatrix} a_{11}+\lambda, & a_{12}, & a_{13}, & \&c. \\ a_{21}, & a_{22}+\lambda, & a_{23}, & \&c. \\ a_{31}, & a_{32}, & a_{33}+\lambda, & \&c. \\ \&c. & & & \end{vmatrix},$$

and form from it a minor, as in Art. 37, by erasing the outside line and column; form from this again another minor by the same rule, and so on. We thus have a series of functions of λ, whose degrees regularly diminish from the n^{th} to the 1^{st}; and we may take any positive constant to complete the series. Now, if we substitute successively in this series any two values of λ, and count in each case the variations of sign, as in Sturm's theorem, it is easy to see that the difference in the number of variations cannot exceed the number of roots of the equation of the n^{th} degree which lie between the two assumed values of λ. This appears at once from what was proved in Art. 37,

* It occurs in the determination of the secular inequalities of the planets (see Laplace, *Mécanique Céleste*, Part I., Book II., Art. 56).

that if λ be taken so as to make any of these minors vanish, the two adjacent functions in the series will have opposite signs. It follows, then, precisely as in the proof of Sturm's theorem, that if we diminish λ regularly from $+\infty$ to $-\infty$, when λ passes through a root of any of these minors, the number of variations in the series will not be affected; and that a change in the number of variations can only take place when λ passes through a root of the first equation, namely, that in which λ enters in the n^{th} degree. The total number of variations, therefore, cannot exceed the number of real roots of this equation.

But, obviously, in all these functions the sign of the highest power of λ is positive; hence, when we substitute $+\infty$, we get no variation; when we substitute $-\infty$, the terms become alternately positive and negative, and we get n variations; the equation we are discussing must, therefore, have n real roots. It is easy to see, in like manner, that the roots of each function of the series are all real, and that the roots of each are interposed as limits between the roots of the function next above it in the series.

47. It will be perceived that in the preceding Article we have substituted, for the functions of Sturm's theorem, another series of functions possessing the same fundamental property, viz. that when one vanishes, the two adjacent to it have opposite signs. Borchardt's proof, however, which we now proceed to give, depends on a direct application of Sturm's theorem.

The first principle which it will be necessary to use is a theorem given by Sylvester (*Philosophical Magazine*, December, 1839), that the several functions in Sturm's series, expressed in terms of the roots of the given equation, differ only by positive square multipliers from the following. The first two (namely, the function itself and the first derived function) are, of course, $(x-\alpha)(x-\beta)(x-\gamma)$ &c., $\Sigma(x-\beta)(x-\gamma)$ &c.; and the remaining ones are

$$\Sigma(\alpha-\beta)^2(x-\gamma)(x-\delta)\text{&c.};\ \Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)\text{&c.,&c.,}$$

where we take the product of any k factors of the given

equation, and multiplying by the product of the squares of the differences of all the roots not contained in these factors, form the corresponding symmetric function. We commence by proving this theorem.*

48. In the first place, let U be the function, V its first derived function, R_2, R_3, &c., the series of Sturm's remainders; then it is easy to see that any one of them can be expressed in the form $AV - BU$. For, from the fundamental equations

$$U = Q_1 V - R_2, \quad V = Q_2 R_2 - R_3, \quad R_2 = Q_3 R_3 - R_4, \text{ \&c.},$$

we have

$$R_2 = Q_1 V - U,$$

$$R_3 = Q_2 R_2 - V = (Q_2 Q_1 - 1) V - Q_2 U,$$

$$R_4 = (Q_3 Q_2 - 1) R_2 - Q_3 V = (Q_1 Q_2 Q_3 - Q_1 - Q_3) V - (Q_2 Q_3 - 1) U,$$

and so on. We have then in general† $R_k = AV - BU$, where, since all the Q's are of the first degree in x, it is easy to see that A is of the degree $k-1$, and B of the degree $k-2$, while R_k is of the degree $n-k$.

But this property would suffice to determine R_2, R_3, &c., directly. Thus, if in the equation $R_2 = Q_1 V - U$, we assume $Q_1 = ax + b$, where a and b are unknown constants, the condition that the coefficients of the highest two powers of x on the right-hand side of the equation must vanish (since R_2 is only of the degree $n-2$) is sufficient to determine a and b. And so in

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* I suppose that Sylvester must have originally divined the form of these functions from the characteristic property of Sturm's functions, viz. that if the equation has two equal roots $\alpha = \beta$, every one of them must become divisible by $x - \alpha$. Consequently, if we express any one of these functions as the sum of a number of products $(x-\alpha)(x-\beta)$ &c., every product which does not include either $x-\alpha$ or $x-\beta$ must be divisible by $(\alpha-\beta)^2$; and it is evident in this way that the theorem *ought* to be true. The method of verification here employed does not differ essentially from Sturm's proof, *Liouville*, vol. VII. p. 356.

† The theory of continued fractions, which we are virtually applying here, shews that if we have $R_k = A_k V - B_k U$, $R_{k+1} = A_{k+1} V - B_{k+1} U$, then $A_k B_{k+1} - A_{k+1} B_k$ is constant and $= 1$. In fact, since $R_{k+1} = Q_k R_k - R_{k-1}$, we have

$$A_{k+1} = Q_k A_k - A_{k-1}, \quad B_{k+1} = Q_k B_k - B_{k-1},$$

whence

$$A_k B_{k+1} - A_{k+1} B_k = A_{k-1} B_k - A_k B_{k-1},$$

and by taking the values in the first two equations above, namely, where $k = 2$ and $k = 3$, we see that the constant value $= 1$.

general, if in the function $AV-BU$ we write for A the most general function of the $(k-1)^{\text{th}}$ degree containing k constants, and for B the most general function of the $(k-2)^{\text{th}}$ degree containing $k-1$ constants, we appear to have in all $2k-1$ constants at our disposal, and have in reality one less, since one of the coefficients may by division be made $=1$.* We have then just constants enough to be able to make the first $2k-2$ terms of the equation vanish, or to reduce it from the degree $n+k-2$ to the degree $n-k$. The problem, then, to form a function of the degree $n-k$, and expressible in the form $AV-BU$, where A and B are of the degrees $k-1$, $k-2$, is perfectly definite, and admits but of one solution. If, then, we have ascertained that any function R_k is expressible in the form $AV-BU$, where A and B are of the right degree, we can infer that R_k must be identical with the corresponding Sturm's remainder, or at least only differ from it by a constant multiplier. It is in this way that we shall identify with Sturm's remainders the expressions in terms of the roots, Art. 47.

49. Let us now, to fix the ideas, take any one of these functions, suppose

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)(x-\varepsilon)\ \&c.,$$

and we shall prove that it is of the form $AV-BU$, where A is of the second degree, and B of the first in x. Now we can immediately see what we are to assume for the form of A, by making $x=\alpha$ on both sides of the equation. The right-hand side of the equation will then become

$$A(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)\ \&c.,$$

since U vanishes; and the left-hand side will become

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\delta)(\alpha-\varepsilon)\ \&c.$$

It follows, then, that the supposition $x=\alpha$ must reduce A to the form $\Sigma(\beta-\gamma)^2(\alpha-\beta)(\alpha-\gamma)$, and it is at once suggested that we ought to take for A the symmetric function

$$\Sigma(\beta-\gamma)^2(x-\beta)(x-\gamma).$$

* Just as the six constants in the most general equation of a conic are only equivalent to five independent constants, and only enable us to make the curve satisfy five conditions.

And in like manner, in the general case, we are to take for A the symmetric function of the product of $k-1$ factors of the original equation multiplied by the product of the squares of the differences of all the roots which enter into these factors. It will not be necessary to our purpose actually to determine the coefficients in B, which we shall therefore leave in its most general form. Let us then write down

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)\,\&c. = \Sigma(\alpha-\beta)^2(x-\alpha)(x-\beta)$$
$$\times\,\Sigma_1(x-\beta)(x-\gamma)\,\&c. + (ax+b)(x-\alpha)(x-\beta)\,\&c.,$$

which we are to prove is an identical equation. Now, since an equation of the p^{th} degree can only have p roots, if such an equation is satisfied by more than p values of x, it must be an identical equation, or one in which the coefficients of the several powers of x separately vanish. But the equation we have written down is satisfied for each of the n values $x=\alpha$, $x=\beta$, &c., no matter what the values of a and b may be. And if we substitute any other two values of x, then, by solving for a and b from the equations so obtained, we can determine a and b, so that the equation may be satisfied for these two values. It is, therefore, satisfied for $n+2$ values of x, and since it is only an equation of the $(n+1)^{th}$ degree, it must be an identical equation. And the corresponding equation in general, which is of the degree $n+k-1$, is satisfied immediately for any of the n values $x=\alpha$, &c.; while B being of the degree $k-1$ we can determine the k constants which occur in its general expression, so that the equation may be satisfied for k other values; the equation is, therefore, an identical equation.

50. We have now proved that the functions written in Art. 48 being of the form $AV-BU$ are either identical with Sturm's remainders, or only differ from them by constant factors. It remains to find out the value of these factors, which is an essential matter, since it is on the signs of the functions that everything turns. Calling Sturm's remainders, as before, R_2, R_3, &c., let Sylvester's forms (Art. 47) be T_2, T_3, &c., then we have proved that the latter are of the form $T_2=\lambda_2 R_2$, $T_3=\lambda_3 R_3$, &c., and we want to determine λ_2, λ_3, &c. We can

at once determine λ_2 by comparing the coefficients of the highest powers of x on both sides of the identity $T_2 = A_2 V - B_2 U$; for x^n does not occur in T_2, while in V the coefficient of x^{n-1} is n, and the coefficient of x is also n in A_2, which $= \Sigma(x-\alpha)$; hence $B_2 = n^2$. But the equation $T_2 = A_2 V - B_2 U$ must be identical with the equation $R_2 = Q_1 V - U$ multiplied by λ_2; we have, therefore, $\lambda_2 = n^2$.

To determine in general λ_k it is to be observed that since any equation $T_k = A_k V - B_k U$ is λ_k times the corresponding equation for R_k, and since in the latter case it was proved (note, Art. 48) that $A_k B_{k+1} - A_{k+1} B_k = 1$, the corresponding quantity for T_k, T_{k+1} must $= \lambda_k \lambda_{k+1}$. Now from the equations

$$T_k = A_k V - B_k U, \quad T_{k+1} = A_{k+1} V - B_{k+1} U,$$

we have

$$A_{k+1} T_k - A_k T_{k+1} = (A_k B_{k+1} - A_{k+1} B_k)\, U = \lambda_k \lambda_{k+1} U.$$

Comparing the coefficients of the highest powers of x on both sides of the equation, and observing that the highest power does not occur in $A_k T_{k+1}$, we have the product of the leading coefficients of A_{k+1} and $T_k = \lambda_k \lambda_{k+1}$. But if we write

$$\Sigma(\alpha-\beta)^2 = p_2, \quad \Sigma(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2 = p_3, \text{ \&c.},$$

we have, on inspection of the values in Arts. 47, 49, the leading coefficient in $T_2 = p_2$, in $T_3 = p_3$, &c., and in $A_2 = n$, in $A_3 = p_2$, in $A_4 = p_3$, &c. Hence

$$p_2^2 = \lambda_2 \lambda_3,\ p_3^2 = \lambda_3 \lambda_4,\ p_4^2 = \lambda_4 \lambda_5, \text{ \&c., whence } \lambda_3 = \frac{p_2^2}{n^2},\ \lambda_4 = \frac{n^2 p_3^2}{p_2^2}, \text{ \&c.}$$

The important matter, then, is that these coefficients are all positive squares, and therefore, as in using Sturm's theorem we are only concerned with the signs of the functions, we may omit them altogether.

51. When we want to know the total number of imaginary roots of an equation, it is well known that we are only concerned with the coefficients of the highest powers of x in Sturm's functions, there being as many pairs of imaginary roots as there are variations in the signs of these leading terms. And since the signs of the leading terms of T_2, T_3, &c. are the same as those of R_2, R_3, &c., it follows that an equation has as

many pairs of imaginary roots as there are variations in the series of signs of $1, n, \Sigma(\alpha-\beta)^2, \Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$, &c. This theorem may be stated in a different form by means of Ex. 3, Art. 26, and we learn that an equation has as many pairs of imaginary roots as there are variations in the signs of the series of determinants

$$1,\ s_0,\ \begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix},\ \begin{vmatrix} s_0, & s_1, & s_2 \\ s_1, & s_2, & s_3 \\ s_2, & s_3, & s_4 \end{vmatrix},\ \begin{vmatrix} s_0, & s_1, & s_2, & s_3 \\ s_1, & s_2, & s_3, & s_4 \\ s_2, & s_3, & s_4, & s_5 \\ s_3, & s_4, & s_5, & s_6 \end{vmatrix},\ \text{&c.},$$

the last in the series being the discriminant; and the condition that the roots of an equation should be all real is simply that every one of these determinants should be positive.

52. We return now, from this digression on Sturm's theorem, to Borchardt's proof, of which we commenced to give an account, Art. 47; and it is evident that in order to apply the test just obtained, to prove the reality of the roots of the equation got by expanding the determinant of Art. 46, it will be first necessary to form the sums of the powers of the roots of that equation. For the sake of brevity, we confine our proof to the determinant of the third order, it being understood that precisely the same process applies in general; and, for convenience, we change the sign of λ, which will not affect the question as to the *reality* of its values. Then it appears immediately, on expanding the determinant, that $s_1 = a_{11} + a_{22} + a_{33}$, since the determinant is of the form $\lambda^3 - \lambda^2(a_{11} + a_{22} + a_{33}) + \text{&c.}$ And in the general case s_1 is equal to the sum of the leading constituents. We can calculate s_2 as follows: The determinant may be supposed to have been derived by eliminating x, y, z between the equations

$$\lambda x = a_{11}x + a_{12}y + a_{13}z,\quad \lambda y = a_{21}x + a_{22}y + a_{23}z,\quad \lambda z = a_{31}x + a_{32}y + a_{33}z.$$

Multiply each of these equations by λ, and substitute on the right-hand side for $\lambda x, \lambda y, \lambda z$ their values, thus we get

$$\lambda^2 x = (a_{11}^2 + a_{12}^2 + a_{13}^2)x + (a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23})y + (a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33})z,$$
$$\lambda^2 y = (a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13})x + (a_{21}^2 + a_{22}^2 + a_{23}^2)y + (a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33})z,$$
$$\lambda^2 z = (a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13})x + (a_{31}a_{21} + a_{32}a_{22} + a_{33}a_{23})y + (a_{31}^2 + a_{32}^2 + a_{33}^2)z,$$

from these eliminating x, y, z, we have a determinant of form exactly similar to that which we are discussing, and which may be written

$$\begin{vmatrix} b_{11}-\lambda^2, & b_{12}, & b_{13} \\ b_{21}, & b_{22}-\lambda^2, & b_{23} \\ b_{31}, & b_{32}, & b_{33}-\lambda^2 \end{vmatrix}.$$

Then, of course in like manner,

$$s_2 = b_{11}+b_{22}+b_{33} = a_{11}^{\ 2}+a_{22}^{\ 2}+a_{33}^{\ 2}+2a_{12}^{\ 2}+2a_{23}^{\ 2}+2a_{31}^{\ 2}.$$

The same process applies in general and enables us from s_p to compute s_{p+1}. Thus suppose we have got the system of equations

$$\lambda^p x = d_{11}x+d_{12}y+d_{13}z, \quad \lambda^p y = d_{21}x+d_{22}y+d_{23}z, \quad \lambda^p z = d_{31}x+d_{32}y+d_{33}z,$$

from which we could deduce, as above, $s_p = d_{11}+d_{22}+d_{33}$; then multiplying both sides by λ, and substituting for λx, &c. their values, we get

$$\begin{aligned}
\lambda^{p+1}x &= (d_{11}a_{11}+d_{12}a_{12}+d_{13}a_{13})\,x+(d_{11}a_{21}+d_{12}a_{22}+d_{13}a_{23})\,y \\
&\qquad +(d_{11}a_{31}+d_{12}a_{32}+d_{13}a_{33})\,z, \\
\lambda^{p+1}y &= (d_{21}a_{11}+d_{22}a_{12}+d_{23}a_{13})\,x+(d_{21}a_{21}+d_{22}a_{22}+d_{23}a_{23})\,y \\
&\qquad +(d_{21}a_{31}+d_{22}a_{32}+d_{23}a_{33})\,z, \\
\lambda^{p+1}z &= (d_{31}a_{11}+d_{32}a_{12}+d_{33}a_{13})\,x+(d_{31}a_{21}+d_{32}a_{22}+d_{33}a_{23})\,y \\
&\qquad +(d_{31}a_{31}+d_{32}a_{32}+d_{33}a_{33})\,z,
\end{aligned}$$

whence $s_{p+1} = d_{11}a_{11}+d_{22}a_{22}+d_{33}a_{33}+2d_{12}a_{12}+2d_{23}a_{23}+2d_{31}a_{31}$.

53. We shall now shew, by the help of these values for s_p, &c., that each of the determinants at the end of Art. 51 can be expressed as the sum of a number of squares, and is therefore essentially positive.* Thus write down the set of constituents

$$\left\| \begin{matrix} 1, & 1, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ a_{11}, & a_{22}, & a_{33}, & a_{23}, & a_{31}, & a_{12}, & a_{23}, & a_{31}, & a_{12} \end{matrix} \right\|,$$

then it is easy to see that $\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix}$ is the determinant formed

* M. Kummer first found out by actual trial that the discriminant of the cubic which determines the axes of a surface of the second degree is resolvable into a sum of squares. (*Crelle*, vol. XXVI., p. 268). The general theory given here is due, as we have said, to Borchardt.

from this, by the method of Art. 24, which expresses it as the sum of all possible squares of determinants which can be formed by taking any two of the nine columns written above. The determinant $\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix}$ is thus seen to be resolvable into the sum of the squares

$$(a_{11}-a_{22})^2+(a_{22}-a_{33})^2+(a_{33}-a_{11})^2+6\,(a_{23}^{\,2}+a_{31}^{\,2}+a_{12}^{\,2}),$$

and is therefore essentially positive. Again, if we write down

$$\left\| \begin{matrix} 1, & 1, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ a_{11}, & a_{22}, & a_{33}, & a_{23}, & a_{31}, & a_{12}, & a_{23}, & a_{31}, & a_{12} \\ b_{11}, & b_{22}, & b_{33}, & b_{23}, & b_{31}, & b_{12}, & b_{23}, & b_{31}, & b_{12} \end{matrix} \right\|,$$

where b_{11}, &c. have the meaning already explained, it will be easily seen from the values we have found that $\begin{vmatrix} s_0, & s_1, & s_2 \\ s_1, & s_2, & s_3 \\ s_2, & s_3, & s_4 \end{vmatrix}$ is the determinant which, in like manner, is equal to the sum of the squares of all possible determinants which can be formed out of the above matrix. And similarly in general.

LESSON VII.

SYMMETRIC FUNCTIONS.

54. We assume the reader to be acquainted with the theory of the symmetric functions of roots of equations as usually given in works on the Theory of Equations. Thus, we suppose him to be acquainted with Newton's formulæ for calculating the sums of the powers of the roots of the equation

$$x^n-p_1x^{n-1}+p_2x^{n-2}-p_3x^{n-3}+\&c.=0,$$

viz. $s_1 - p_1 = 0$, $s_2 - p_1 s_1 + 2p_2 = 0$, $s_3 - p_1 s_2 + p_2 s_1 - 3p_3 = 0$, &c.,

whence $s_1 = p_1$, $s_2 = p_1^2 - 2p_2$, $s_3 = p_1^3 - 3p_1 p_2 + 3p_3$, &c.,

and with the formulæ

$$\Sigma\alpha^m\beta^p = s_m s_p - s_{m+p},$$

$$\Sigma\alpha^m\beta^p\gamma^q = s_m s_p s_q - s_{m+p} s_q - s_{m+q} s_p - s_{p+q} s_m + 2s_{m+p+q}, \text{ &c.}$$

We can thus calculate $\Sigma\alpha^m\beta^p$, &c. first in terms of the sums s_1, s_2, &c., and ultimately in terms of the coefficients p_1, p_2, &c.*

Ex. We can get determinant expressions for the sums of powers in terms of the coefficients, or *vice versâ*, by solving, as in Art. 29, the system of linear equations above written, for s_1, s_2, &c., or p_1, p_2, &c. Thus we have

$$s_2 = \begin{vmatrix} p_1, & 1 \\ 2p_2, & p_1 \end{vmatrix}; \quad s_3 = \begin{vmatrix} p_1, & 1, & 0 \\ 2p_2, & p_1, & 1 \\ 3p_3, & p_2, & p_1 \end{vmatrix}; \quad s_4 = \begin{vmatrix} p_1, & 1, & 0, & 0 \\ 2p_2, & p_1, & 1, & 0 \\ 3p_3, & p_2, & p_1, & 1 \\ 4p_4, & p_3, & p_2, & p_1 \end{vmatrix}; \text{ &c.}$$

$$1.2.p_2 = \begin{vmatrix} s_1, & 1 \\ s_2, & s_1 \end{vmatrix}; \quad 1.2.3.p_3 = \begin{vmatrix} s_1, & 1, & 0 \\ s_2, & s_1, & 2 \\ s_3, & s_2, & s_1 \end{vmatrix}, \quad 1.2.3.4.p_4 = \begin{vmatrix} s_1, & 1, & 0, & 0 \\ s_2, & s_1, & 2, & 0 \\ s_3, & s_2, & s_1, & 3 \\ s_4, & s_3, & s_2, & s_1 \end{vmatrix}; \text{ &c.}$$

55. It is more natural to start from the equations

$$\Sigma\alpha = p_1, \quad \Sigma\alpha\beta = p_2, \quad \Sigma\alpha\beta\gamma = p_3, \text{ &c.},$$

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* This process is a very bad one, in fact if it were employed throughout, we should have for instance to calculate $\Sigma\alpha\beta\gamma$, that is p_3 from the formula

$$\begin{aligned} 6\Sigma\alpha\beta\gamma = \quad & s_1^3 \quad = p_1^3 \\ & - 3s_1 s_2 \quad - 3p_1 (p_1^2 - 2p_2) \\ & + 2s_3 \quad + 2 (p_1^3 - 3p_1 p_2 + 3p_3) \\ & \quad = 6p_3, \end{aligned}$$

but the process introduces terms p_1^3 and $p_1 p_2$ each of a higher order than p_3 (reckoning the order of each coefficient as unity) with numerical coefficients which destroy each other. And so again $\Sigma\alpha^2\beta$ would be calculated from the formula

$$\begin{aligned} \Sigma\alpha^2\beta = \quad & s_1 s_2 = \quad p_1 (p_1^2 - 2p_2) \\ & - s_3 \quad - (p_1^3 - 3p_1 p_2 + 3p_3) \\ & \quad = p_1 p_2 - 3p_3, \end{aligned}$$

but there is here also a term p_1^3 of a higher order, with numerical coefficients which destroy each other. And the order in which the several expressions are derived, the one from the other is a non-natural one; s_3 is required for the determination of $\Sigma\alpha^2\beta$, whereas it is properly $\Sigma\alpha^2\beta$ which leads to the value of s_3.

and thence derive the sets of equations

$$\begin{aligned}
p_1 &= \Sigma\alpha, \\
p_2 &= \Sigma\alpha\beta, \\
p_1^2 &= \Sigma\alpha^2 + 2\Sigma\alpha\beta, \\
p_3 &= \Sigma\alpha\beta\gamma, \\
p_1p_2 &= \Sigma\alpha^2\beta + 3\Sigma\alpha\beta\gamma, \\
p_1^3 &= \Sigma\alpha^3 + 3\Sigma\alpha^2\beta + 6\Sigma\alpha\beta\gamma;
\end{aligned}$$

we thus have 1 equation to give $\Sigma\alpha$; 2 equations to give $\Sigma\alpha\beta$ and $\Sigma\alpha^2$; 3 equations to give $\Sigma\alpha\beta\gamma$, $\Sigma\alpha^2\beta$, $\Sigma\alpha^3$, and so on. And taking, for instance, the third set of equations, the first equation gives $\Sigma\alpha\beta\gamma$, the second then gives $\Sigma\alpha^2\beta$, and the third then gives $\Sigma\alpha^3$; thus

$$\begin{aligned}
\Sigma\alpha\beta\gamma &= p_3, \\
\Sigma\alpha^2\beta &= p_1p_2 - 3p_3, \\
\Sigma\alpha^3 &= p_1^3 - 3(p_1p_2 - 3p_3) - 6p_3, \\
&= p_1^3 - 3p_1p_2 + 3p_3.
\end{aligned}$$

The process for the formation of the successive sets of equations is further explained and developed in Prof. Cayley's "Memoir on the Symmetric Functions of the Roots of an Equation," *Phil. Trans.*, vol. CXLVII, (1857), and the original and inverse sets of equations, for equations up to the order 10, are therein exhibited in the form of tables (see Appendix).

56. If we have any homogeneous function of the coefficients p_1, p_2, &c., we shall use the word *order* of that function, in the usual sense, to denote the number of factors of which each term consists. Thus, if any term were $p_1^r p_2^s p_3^t$, the order of the function would be $r+s+t$. If the function be not homogeneous, the order of the function is as usual regulated by the order of the highest term. By the *weight* of a function we shall understand the sum of the suffixes attached to each factor. Thus, if any term were $p_1^r p_2^s p_3^t$, the *weight* of the function would be $r+2s+3t$; or, again, if any term were $p_r p_s p_t$, this term would be of the third order, while its weight would be $r+s+t$. In the case of every function, with which we shall be concerned, the weight will be the same for every term.

57. On inspecting the expressions given above for s_1, s_2, s_3, &c. in terms of the coefficients, it is obvious that the weight of every term in s_2 is two, in s_3 is three, and it is easy to conclude by induction that the weight of every term in s_n is n. In like manner, it is evident that the weight of $\Sigma\alpha^m\beta^p$ is $m+p$, of $\Sigma\alpha^m\beta^p\gamma^q$ is $m+p+q$, &c.

This may be proved in general as follows: If for every root α, β, γ, &c. we substitute λ times α, λ times β, λ times γ, &c., we evidently multiply the function $\Sigma\alpha^m\beta^p\gamma^q$ by λ^{m+p+q}. But it is known that if we multiply every root by λ, we multiply p_1 by λ, p_2 by λ^2, p_3 by λ^3, &c. It follows then that $\Sigma\alpha^m\beta^p\gamma^q$ expressed in terms of the coefficients must be such that if we substitute for p_1, λp_1, for p_2, $\lambda^2 p_2$, and so on, we shall multiply every term by λ^{m+p+q}; and this, in other words, is saying that the weight of every term is $m+p+q$.

58. Since

$p_1=\alpha+\beta+\gamma+$ &c., $p_2=\alpha(\beta+\gamma+$ &c.$)+\beta\gamma+$ &c., &c.,

and none of the coefficients, p_3, p_4, &c. contains any power of α beyond the first, it is plain that the order of any symmetric function $\Sigma\alpha^m\beta^p\gamma^q$ (where m is supposed to be greater than p or q) must be at least m. For, of course, unless there are at least m factors, each containing α, α^m cannot appear in the product. But, conversely, any symmetric function, whose order is m, will contain some terms involving α^m. For if q_1, q_2, q_3, &c. be the sum, sum of products in pairs, in threes, &c. of β, γ, δ, &c., we have $p_1=\alpha+q_1$, $p_2=\alpha q_1+q_2$, $p_3=\alpha q_2+q_3$, &c., and the coefficient of the highest power of α in such a term as $p_2^r p_3^s p_4^t$, will be $q_1^r q_2^s q_3^t$; and, conversely, the multiplier $q_1^r q_2^s q_3^t$ can *only* arise from the term $p_2^r p_3^s p_4^t$. It therefore cannot be made to vanish by the addition of other terms. It follows then that the order of any symmetric function $\Sigma\alpha^m\beta^p\gamma^q$ is equal to the greatest of the numbers m, p, q; for we have proved that it cannot be less than that number, and that it cannot be greater, since functions of a higher order would contain higher powers of α than α^m.

By the help of the two principles just proved (that the weight is the degree in the roots, and the order the highest

degree in any one root), we can write down the literal part of any symmetric function, and it only remains to determine the coefficients. Thus if it were required to form $\Sigma\alpha^2(\beta-\gamma)^2$, we see on inspection that this is a function whose weight is four, and that it is of the second order; that is to say, there cannot be more than two factors in any term. The only terms then that can enter into such a function are p_4, p_3p_1, p_2^2, and the calculation would be complete if we knew with what coefficients these terms are to be affected.*

59. Symmetric functions of the differences of the roots of equations† being those with which we shall have most to deal, it may not be amiss to give a theorem by which the sum of any powers of the differences can be expressed in terms of the sums of the powers of the roots of the given equation. Expanding $(x-\alpha)^m$ by the binomial theorem, and adding the similar expansions for $(x-\beta)^m$, &c., we have at once

$$\Sigma(x-\alpha)^m = s_0x^m - ms_1x^{m-1} + \tfrac{1}{2}m(m-1)s_2x^{m-2} - \&c.$$

Now if we substitute α for x in $\Sigma(x-\alpha)^m$ it becomes $(\alpha-\beta)^m+(\alpha-\gamma)^m+\&c.$; similarly if we substitute β for x it becomes $(\beta-\alpha)^m+(\beta-\gamma)^m+\&c.$, and so on; and when we add the results of all these substitutions, if m be odd, the sum vanishes, since the terms $(\alpha-\beta)^m$, $(\beta-\alpha)^m$ cancel each other. If m be even, the result is $2\Sigma(\alpha-\beta)^m$. But when the same substitutions are made on the right-hand side of the equation last written, and the results added together, we get

$$s_0s_m - ms_1s_{m-1} + \tfrac{1}{2}m(m-1)s_2s_{m-2} - \&c.$$

If m be odd, the last term will be $-s_ms_0$, which will cancel the first term, and, in like manner, all the other terms will destroy each other. But if m be even, the last term will be identical with the first, and so on, and the equation will be divisible by two. Thus, then, when m is even, we have

$$\Sigma(\alpha-\beta)^m = s_0s_m - ms_1s_{m-1} + \tfrac{1}{2}m(m-1)s_2s_{m-2} - \&c.,$$

* The foregoing example of the calculation of $\Sigma\alpha\beta\gamma$, $\Sigma\alpha^2\beta$, $\Sigma\alpha^3$, in effect shows how we can in every case obtain for the determination of the coefficients the required number of linear relations.

† Such functions have been called *critical* functions.

UNIVERSITY OF CALIFORNIA

where the coefficients are those of the binomial until we come to the middle term with which we stop, and which must be divided by two. Thus

$$\Sigma(\alpha-\beta)^4 = s_0 s_4 - 4s_1 s_3 + 3s_2^2, \quad \Sigma(\alpha-\beta)^6 = s_0 s_6 - 6s_1 s_5 + 15 s_2 s_4 - 10 s_3^2, \text{ \&c.}$$

60. Any function of the differences will of course be unchanged if we increase or diminish all the roots by the same quantities, as, for instance, if we substitute $x - \lambda$ for x in the given equation. It then becomes

$$x^n - (p_1 + n\lambda) x^{n-1} + \{p_2 + (n-1)\lambda p_1 + \tfrac{1}{2} n (n-1) \lambda^2\} x^{n-2} - \{p_3 + (n-2)\lambda p_2 + \text{\&c.}\} x^{n-3} + \text{\&c.} = 0.$$

Now any function ϕ of the coefficients p_1, p_2, &c. will, when we alter p_1 into $p_1 + \delta p_1$, p_2 into $p_2 + \delta p_2$, &c., become

$$\phi + \left\{\frac{d\phi}{dp_1}\delta p_1 + \frac{d\phi}{dp_2}\delta p_2 + \text{\&c.}\right\} + \frac{1}{1.2}\left\{\frac{d^2\phi}{dp_1^2}(\delta p_1)^2 + \text{\&c.}\right\} + \text{\&c.}$$

It then, in any function of p_1, p_2, &c., we substitute $p_1 + n\lambda$ for p_1, $p_2 + (n-1)\lambda p_1 + \frac{1}{2} n (n-1) \lambda^2$ for p_2, &c., and arrange the result according to the powers of λ, it becomes

$$\phi + \lambda\left\{n\frac{d\phi}{dp_1} + (n-1) p_1 \frac{d\phi}{dp_2} + (n-2) p_2 \frac{d\phi}{dp_3} + \text{\&c.}\right\} + \lambda^2(\text{\&c.}) = 0.$$

But since we have seen that any function of the differences is unchanged by the substitution, no matter how small λ be, it is necessary that any function of the differences, when expressed in terms of the coefficients, should satisfy the differential equation

$$n\frac{d\phi}{dp_1} + (n-1) p_1 \frac{d\phi}{dp_2} + (n-2) p_2 \frac{d\phi}{dp_3} + \text{\&c.} = 0.$$

Ex. 1. Let it be required to form $\Sigma(\alpha - \beta)^2$. We know that its order and weight are both $= 2$. It must therefore be of the form $Ap_2 + Bp_1^2$. Applying the differential equation, we have $\{(n-1) A + 2nB\} p_1 = 0$, whence B is proportional to $n - 1$ and A to $-2n$. The function then can only differ by a factor from $(n-1) p_1^2 - 2np_2$.

The factor may be shewn to be unity by supposing $\alpha = 1$ and all the other roots $= 0$, when $p_1 = 1$, $p_2 = 0$, and the value just written reduces to $n - 1$, as it ought to do.

Ex. 2. To form for a cubic the product of the squares of the differences $(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2$. This is a function of the order 4 and weight 6. It must

therefore be of the form

$$Ap_3^2 + Bp_3p_2p_1 + Cp_3p_1^3 + Dp_2^3 + Ep_2^2p_1^2.$$

Operating with $3\frac{d}{dp_1} + 2p_1\frac{d}{dp_2} + p_2\frac{d}{dp_3}$ it becomes

$$(2A + 3B)\, p_3p_2 + (2B + 9C)\, p_3p_1^2 + (B + 6D + 6E)\, p_2^2p_1 + (C + 4E)\, p_2p_1^3,$$

and as this is to vanish identically, we must have $C = -4E$, $B = 18E$, $A = -27E$, $D = -4E$, or the function can only differ by a factor from

$$p_1^2p_2^2 + 18p_1p_2p_3 - 4p_2^3 - 4p_3p_1^3 - 27p_3^2.$$

The factor may be shewn to be unity by supposing γ and consequently p_3 to be $= 0$.

61. We shall in future usually employ homogeneous equations. Thus, writing $x : y$ for x, and clearing of fractions, the equation we have used becomes

$$x^n - p_1x^{n-1}y + p_2x^{n-2}y^2 \ldots \pm p_ny^n = 0.$$

We give x^n a coefficient for the sake of symmetry; and we find it convenient to give the terms the same coefficients as in the binomial theorem; and so write the equation

$$a_0x^n + na_1x^{n-1}y + \tfrac{1}{2}n\,(n-1)\, a_2x^{n-2}y^2 + \ldots + na_{n-1}xy^{n-1} + a_ny^n = 0,$$

or, as this may be for shortness represented,

$$(a_0, a_1, \ldots a_n)\,(x, y)^n = 0.$$

One advantage of using the binomial coefficients is, that thus all functions of the differences of the roots will, when expressed in terms of the coefficients, be such that the sum of the numerical coefficients will be nothing. For we get the sum of the numerical coefficients by making $a_0 = a_1 = a_2 = \&c. = 1$; but on this supposition all the roots of the original equation become equal, and all the differences vanish.

When we speak of a symmetric function of the roots of the homogeneous equation, we understand that the equation having been divided by a_0y^n, the corresponding symmetric function has been formed of the coefficients $\frac{a_1}{a_0}$, $\frac{a_2}{a_0}$, &c. of the equation in $\frac{x}{y}$, and that it has been cleared of fractions by multiplying by the highest power of a_0 in any denominator. In this way, every symmetric function will be a homogeneous function of the coefficients a_0, a_1, &c.; for, before it was cleared of fractions, it was

a homogeneous function of the degree 0, and it remains homogeneous when every term is multiplied by the same quantity. Or we may state the theory for the symmetric functions of the roots of the homogeneous equation, without first transforming it to an equation in $x:y$. If one of the roots of the latter equation be α, that is, if a factor of the function is $x-\alpha y$, then it is evident that the homogeneous equation is satisfied by any system of values x', y' for which we have $x'=\alpha y'$, since it is manifest that we are only concerned with the ratio $x':y'$. And since the equation divided by y^n is resolvable into factors, so the homogeneous equation is plainly reducible to a product of factors $(xy'-yx')\,(xy''-yx'')\,(xy'''-yx''')$, &c. Actually multiplying and comparing with the original equation, we get

$$a_0=y'y''y'''\text{ &c.},\quad na_1=-\Sigma x'y''y'''\text{ &c.},\quad \tfrac{1}{2}n(n-1)\,a_2=\Sigma x'x''y'''\text{ &c.},$$

$$a_n=\pm x'x''x'''\text{ &c.},\quad na_{n-1}=\mp\Sigma y'x''x'''\text{ &c.}$$

By making all the y's $=1$, these expressions become the ordinary expressions for the coefficients of an equation in terms of its roots, x', x'', &c. And conversely, any symmetric function expressed in the ordinary way in terms of the roots x', x'', may be reduced to the other form, by imagining each x' divided by the corresponding y', and then the whole multiplied by such a power of $y'y''$ &c., as will clear it of fractions. Thus the sum of the squares of the differences $\Sigma(x'-x'')^2$ becomes $\Sigma(x'y''-y'x'')^2y'''^2y'^2$ &c. And generally any function of the differences will consist of the sum of products of determinants of the form $(x'y''-y'x'')\,(x'y'''-y'x''')$ &c., by powers of y', y'', &c.

62. The differential equation which we have given for functions of the differences of the roots requires to be modified when the equation has been written with binomial coefficients. Thus, if in the equation $a_0x^n+na_1x^{n-1}y+\text{&c.}=0$, we write $x+\lambda y$ for x, the new a_1 becomes $a_1+\lambda a_0$, a_2 becomes $a_2+2\lambda a_1+\lambda^2a_0$, a_3 becomes $a_3+3\lambda a_2+3\lambda^2a_1+\lambda^3a_0$, &c., and any function ϕ of the coefficients is altered by this substitution into

$$\phi+\lambda\left(a_0\frac{d\phi}{da_1}+2a_1\frac{d\phi}{da_2}+3a_2\frac{d\phi}{da_3}+\text{&c.}\right)+\text{&c.}$$

Any function then of the differences, since it remains unaltered when $x+\lambda$ is substituted for x, must satisfy the equation

$$a_0 \frac{d\phi}{da_1} + 2a_1 \frac{d\phi}{da_2} + 3a_2 \frac{d\phi}{da_3} + \&c. = 0.$$

In like manner any function which remains unaltered, when $y+\lambda$ is substituted for y, must satisfy the equation

$$na_1 \frac{d\phi}{da_0} + (n-1)\, a_2 \frac{d\phi}{da_1} + (n-2)\, a_3 \frac{d\phi}{da_2} + \&c. = 0.$$

Functions of the latter kind are functions of the differences of the reciprocals of the roots, and in the homogeneous notation consist of products of determinants of the form $x'y'' - y'x''$, &c., by powers of x', x'', &c. Functions of the determinants $x'y'' - y'x''$ alone, and not multiplied by any powers of the x's or the y's, will satisfy both the differential equations.

63. It is to be observed that the condition

$$a_0 \frac{d\phi}{da_1} + 2a_1 \frac{d\phi}{da_2} + 3a_2 \frac{d\phi}{da_3} + \&c. = 0$$

is not only necessary but sufficient, in order that ϕ should be unaltered by the transformation $x+\lambda$ for x. We have seen that the coefficient of λ in the transformed equation then vanishes, and the coefficient of λ^2 is easily found to be

$$a_0 \frac{d\phi}{da_2} + 3a_1 \frac{d\phi}{da_3} + 6a_2 \frac{d\phi}{da_4} + \&c. + \frac{1}{1.2}\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right)^2 \phi,$$

where, considering the second term as denoting $\frac{1}{1.2}\Delta.\Delta\phi$, the a_0, a_1, &c. which appear explicitly in $\Delta\phi$ are not to be differentiated. But this being so, the two terms together are $= \frac{1}{1.2}\Delta.\Delta\phi$, where $\Delta.\Delta\phi$ denotes now the complete effect of the operation Δ upon $\Delta\phi$. For, when we operate with the symbol on itself, the result will be the sum of the terms got by differentiating the a_1, a_2, &c., which appear explicitly, together with the result on the supposition that these a_1, a_2, &c., are constant. Thus, then $\Delta\phi$ vanishing identically, we have $\Delta.\Delta\phi = 0$, or the coefficient of λ^2 vanishes. So, in like manner, for the coefficients of the other powers of λ.

Ex. To form for the cubic $a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$, the function

$$\Sigma\,(x_1y_2 - x_2y_1)^2\,(x_2y_3 - x_3y_2)^2\,(x_3y_1 - x_1y_3)^2.$$

This can be derived from Ex. 2, Art. 60, or else directly as follows. The function is to be of the order 4 and weight 6. It must therefore be of the form

$$Aa_3a_3a_0a_0 + Ba_3a_2a_1a_0 + Ca_3a_1a_1a_1 + Da_2a_2a_2a_0 + Ea_2a_2a_1a_1.$$

Operate with $a_0\dfrac{d}{da_1} + 2a_1\dfrac{d}{da_2} + 3a_2\dfrac{d}{da_3}$, and we get

$$(B+6A)\,a_3a_2a_0a_0 + (3C+2B)\,a_3a_1a_0a_0 + (2E+6D+3B)\,a_2a_2a_1a_0 + (4E+3C)\,a_2a_1a_1a_1 = 0.$$

Equating separately to 0 the coefficient of each term, and taking $A = 1$, we find

$$B = -6,\quad C = 4,\quad D = 4,\quad E = -3.$$

64. M. Serret writes the operation $a_0\dfrac{d\phi}{da_1}$ + &c. in a compact form, which is sometimes convenient. If we imagine a fictitious variable ζ, of which the coefficients a_0, a_1, &c., are such functions, that

$$\frac{da_1}{d\zeta} = a_0,\quad \frac{da_2}{d\zeta} = 2a_1,\quad \frac{da_3}{d\zeta} = 3a_2,\ \&c.,$$

then evidently $\dfrac{d\phi}{d\zeta} = a_0\dfrac{d\phi}{da_1} + 2a_1\dfrac{d\phi}{da_2} + 3a_2\dfrac{d\phi}{da_3}$ + &c.

In like manner $na_1\dfrac{d\phi}{da_0}$ + &c. may be briefly written $\dfrac{d\phi}{d\eta}$, where η is a variable, of which a_0, a_1, &c. are supposed to be such functions, that

$$\frac{da_0}{d\eta} = na_1,\quad \frac{da_1}{d\eta} = (n-1)\,a_2,\ \&c.$$

65. The above operators

$$a_0\frac{d}{da_1} + 2a_1\frac{d}{da_2} + \ldots,\quad na_1\frac{d}{da_0} + (n-1)\,a_2\frac{d}{da_1} + \ldots,$$

may be represented by

$$\{y\delta_x\},\quad \{x\delta_y\}$$

respectively, since the first of them operating on $(a_0, a_1 \ldots a_n)(x, y)^n$ produces the same effect as $y\delta_x$, the second the same effect as $x\delta_y$. If the function be expressed in terms of its roots

$$= a_0(x - \alpha y)\,(x - \beta y)\ldots,$$

then the two operators may be transformed into symbols operating on the roots, as we have

$$-\{y\delta_x\} = \delta_\alpha + \delta_\beta + \ldots,$$

$$\{x\delta_y\} = -(\alpha + \beta + \ldots)a_0\delta_{a_0} + \alpha^2\delta_\alpha + \beta^2\delta_\beta + \ldots,$$

the proof of which may be supplied without difficulty.

LESSON VIII.

ORDER AND WEIGHT OF ELIMINANTS.

66. When we are given k homogeneous equations in k variables (or, what comes to the same thing, k non-homogeneous equations in $k-1$ variables) it is always possible so to combine the equations as to obtain from them a single equation $\Delta = 0$, in which these variables do not appear. We are then said to have eliminated the variables, and the quantity Δ is called the *Eliminant** of the system of equations. Let us take the simplest example, that which we have already considered in the first lesson, where we are given two equations of the first degree $ax + b = 0$, $a'x + b' = 0$. If we multiply the first equation by a', and the second by a, and subtract the first equation from the second, we get $ab' - a'b = 0$, and the quantity $ab' - a'b$ is the eliminant of the two equations. Now it will be observed, that we cannot draw the inference $ab' - a'b = 0$ unless the two given equations are supposed to be simultaneous, that is to say, unless it is supposed that both can be satisfied by the same value of x. For, evidently, when we combine two equations $\phi(x) = 0$, $\psi(x) = 0$, and draw such an inference as $l\phi(x) + m\psi(x) = 0$, it is assumed that x means the same thing in both equations. It follows then that $ab' - a'b = 0$ is the condition, that the two equations can be satisfied by the same value of x, as may also be seen immediately by solving both equations for x, and equating the resulting values. And so generally, if we

* Eliminants are also called *resultants*.

are given any number of equations, $U=0$, $V=0$, $W=0$, &c., we may proceed to combine them, and draw an inference such as $lU+mV+nW=0$, only if the variables have the same values in all the equations. And, if by combining the equations, we arrive at a result not containing the variables, this will vanish if the equations can be satisfied by a common system of values of the variables, and not otherwise. Hence for any such system of equations the eliminant may in general be defined as *that function of the coefficients, whose vanishing expresses that the equations can be satisfied by a common system of values of the variables.*

67. We have now to show how elimination can be performed, and what is the nature of the results arrived at. We commence with two equations written in the non-homogeneous form

$$x^m - p_1x^{m-1} + p_2x^{m-2} - \&c. = 0, \text{ or } \phi(x) = 0,$$
$$x^n - q_1x^{n-1} + q_2x^{n-2} - \&c. = 0, \text{ or } \psi(x) = 0.$$

The vanishing of the eliminant of these equations is, as we have seen, the condition that they should have a common root. If this be the case, some one of the roots of the first equation must satisfy the second. Let the roots of the first equation be α, β, γ, &c., and let us substitute these values successively in the second equation, then some one of the results $\psi(\alpha)$, $\psi(\beta)$, &c. must vanish, and therefore the product of all must certainly vanish. But this product is a symmetric function of the roots of the first equation, and therefore can be expressed in terms of its coefficients, in which state it is the eliminant required. The rule then for elimination by this method, is to take the m factors

$$\psi(\alpha) = \alpha^n - q_1\alpha^{n-1} + q_2\alpha^{n-2} - \&c.,$$
$$\psi(\beta) = \beta^n - q_1\beta^{n-1} + q_2\beta^{n-2} - \&c.,$$
$$\psi(\gamma) = \gamma^n - q_1\gamma^{n-1} + q_2\gamma^{n-2} - \&c., \&c.,$$

to multiply all together, and then substitute for the symmetric functions $(\alpha\beta\gamma)^n$, &c., their values in terms of the coefficients of the first equation.

Ex. To eliminate x between $x^2 - p_1x + p_2 = 0$, $x^2 - q_1x + q_2 = 0$. Multiplying $(\alpha^2 - q_1\alpha + q_2)(\beta^2 - q_1\beta + q_2)$, we get

$$\alpha^2\beta^2 - q_1\alpha\beta(\alpha+\beta) + q_2(\alpha^2+\beta^2) + q_1^2\alpha\beta - q_1q_2(\alpha+\beta) + q_2^2;$$

and then substituting $\alpha+\beta=p_1$, $\alpha\beta=p_2$, $\alpha^2+\beta^2=p_1^2-2p_2$, we have

$$p_2^2-p_1p_2q_1+q_2(p_1^2-2p_2)+p_2q_1^2-q_1q_2p_1+q_2^2,$$

or
$$(p_2-q_2)^2+(p_1-q_1)(p_1q_2-p_2q_1),$$

which is the eliminant required.

68. We obtain in this way the same result (or at most results differing only in sign), whether we substitute the roots of the first equation in the second, or those of the second in the first. In other words, if α', β', γ', &c. be the roots of the second equation, the eliminant may be written at pleasure as the continued product of $\phi(\alpha')$, $\phi(\beta')$, $\phi(\gamma')$, &c., or as the product of $\psi(\alpha)$, $\psi(\beta)$, $\psi(\gamma)$, &c. For remembering that $\phi(x)=(x-\alpha)(x-\beta)(x-\gamma)$, &c., the first form is

$$(\alpha'-\alpha)(\alpha'-\beta)(\alpha'-\gamma)\text{ &c. }(\beta'-\alpha)(\beta'-\beta)(\beta'-\gamma)\text{ &c.},$$

and the second is

$$(\alpha-\alpha')(\alpha-\beta')(\alpha-\gamma')\text{ &c. }(\beta-\alpha')(\beta-\beta')(\beta-\gamma')\text{ &c.}$$

In either case we get the product of all possible differences between a root of the first equation and a root of the second; and the two products can at most differ in sign.

69. If the equations had been given in the homogeneous form, with or without binomial coefficients,

$$a_0x^m+ma_1x^{m-1}y+\tfrac{1}{2}m(m-1)a_2x^{m-2}y^2+\text{&c.}=0,$$
$$b_0x^n+nb_1x^{n-1}y+\tfrac{1}{2}n(n-1)b_2x^{n-2}y^2+\text{&c.}=0,$$

we can reduce them to the preceding form by dividing them respectively by a_0y^m, b_0y^n, when we have $p_1=-\dfrac{ma_1}{a_0}$, $q_1=-\dfrac{nb_1}{b_0}$, &c. We substitute then these values for p_1, q_1, &c. in the result obtained by the method of the last article, and then clear of fractions by multiplying by the highest power of a_0, or b_0 in any denominator. Thus the eliminant of $a_0x^2+2a_1xy+a_2y^2$, $b_0x^2+2b_1xy+b_2y^2$, obtained in this manner from the result of Ex., Art. 67, is

$$(a_0b_2-a_2b_0)^2+4(a_1b_0-a_0b_1)(a_1b_2-a_2b_1).$$

It is evident thus that *the eliminant is always a homogeneous function of the coefficients of each equation.* For before we cleared of fractions, it was evidently a homogeneous function of the degree 0, and it remains homogeneous when every term is multiplied by the same quantity.

The same thing may be seen by applying to the equations directly the process of Art. 67. Let the values which satisfy the first equation be $x'y'$, $x''y''$, &c.; then, if the equations have a common factor, some one of these values must satisfy the second equation. We must then multiply together

$$(b_0x'^n + nb_1x'^{n-1}y' + \&c.)\,(b_0x''^n + nb_1x''^{n-1}y'' + \&c.)\,(\&c.),$$

which is a homogeneous function of the coefficients b_0, b_1, ... of the second equation, and of course remains so after substituting for the symmetric functions $(x'x''\ \&c.)^n$ &c. their values in terms of the coefficients of the first equation. And in the same manner the function is homogeneous as regards the coefficients of the first equation.

70. *The eliminant of two equations of the m^{th} and n^{th} orders respectively, is of the n^{th} order in the coefficients of the first equation, and of the m^{th} in the coefficients of the second.*

For it may be written either as the product of m factors $\psi(\alpha)$, $\psi(\beta)$, &c., each containing the coefficients of the second equation in the first degree, or else as the product of n factors $\phi(\alpha')$, $\phi(\beta')$, &c., each containing in the first degree the coefficients of the first equation. Or confining our attention to the form $\psi(\alpha).\psi(\beta).\&c.$ we can see that this form, which obviously contains the coefficients of the second equation in the degree m, contains those of the first in the degree n, since the symmetric functions which occur in it may contain the n^{th}, and no higher, power of any root (Art. 58).

71. *The weight of the eliminant is mn;* that is to say, the sum of the suffixes in every term is constant and $=mn$. For if each of the roots α, β; α', β', &c. be multiplied by the same factor λ, then since each of the mn differences $\alpha - \alpha'$ (see

Art. 68) is multiplied by this factor λ, the eliminant will be multiplied by λ^{mn}. But the roots of the two equations will be multiplied by λ if for p_1, q_1 we substitute λp_1, λq_1; for p_2, q_2; $\lambda^2 p_2$, $\lambda^2 q_2$; &c. We see then, that if we make this substitution in the eliminant, the effect will be that every term will be multiplied by λ^{mn}; or, in other words, the sum of the suffixes in every term will be mn. The same thing may also be seen to follow from the principle of Art. 57. In $\psi(x)$ the sum of the index of every term and the suffix of the corresponding coefficient is n; that is to say, $\psi(x)$ consists of the sum of a number of terms, each of the form $q_{n-i}x^i$. If, then, we take any term at random in each of the factors $\psi(\alpha)$, $\psi(\beta)$, &c., the corresponding term in the product will be $q_{n-i}q_{n-j}q_{n-k}\alpha^i\beta^j\gamma^k$, &c., and if we combine with this all other terms in which the same coefficients of the second equation occur, we get $q_{n-i}q_{n-j}q_{n-k}\Sigma\alpha^i\beta^j\gamma^k$, &c. The sum of the suffixes of the q's is $n-i+n-j+n-k+$&c., or since there are m factors, the sum is $mn-(i+j+k+$&c.). But, by Art. 57, the sum of the suffixes of the p's in the expression for $\Sigma\alpha^i\beta^j\gamma^k$, &c. is $i+j+k+$&c. Therefore the sum of both sets of suffixes is mn, which was to be proved.

The result at which we have arrived may be otherwise stated thus:* *If p_1, q_1 contain any new variable z in the first degree; if p_2, q_2 contain it in the second and lower degrees; if p_3, q_3 in the third, and so on; then the eliminant will in general contain this variable in the mn^{th} degree.*

It is evident that the results of this and of the last article are equally true if the equations had been written in the homogeneous form a_0x^m+&c., because the suffixes in the two forms mutually correspond. And again, from symmetry, it follows that the result of this article would be equally true if the equations had been written in the form $a_m x^m + ma_{m-1}x^{m-1}y+$&c., where the suffix of any coefficient corresponds to the power of x which it multiplies, instead of to the power of y.

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* Or again thus: if in the eliminant we substitute for each coefficient p_α, the term x^α which it multiplies in the original equation, every term of the eliminant will be divisible by x^{mn}. Or, in the homogeneous form, if we substitute for each coefficient a_α the term $x^\alpha y^{m-\alpha}$, which it multiplies, every term of the eliminant will be divisible by $x^{mn}y^{mn}$.

72. Since the eliminant is a function of the differences between a root of one equation and a root of the other, it will be unaltered if the roots of each equation be increased by the same quantity; that is to say, if we substitute $x+\lambda$ for x in each equation. It follows then, as in Art. 60, that the eliminant must satisfy the differential equation

$$m\frac{d\Delta}{dp_1}+(m-1)\,p_1\frac{d\Delta}{dp_2}+\&c.+n\frac{d\Delta}{dq_1}+(n-1)\,q_1\frac{d\Delta}{dq_2}+\&c.=0,$$

or, as in Art. 62, if the equations had been written with binomial coefficients, we have

$$a_0\frac{d\Delta}{da_1}+2a_1\frac{d\Delta}{da_2}+\&c.+b_0\frac{d\Delta}{db_1}+2b_1\frac{d\Delta}{db_2}+\&c.=0.$$

73. *Given two homogeneous equations between three variables, of the m^{th} and n^{th} degrees respectively, the number of systems of values of the variables which can be found to satisfy simultaneously the two equations is mn.**

Let the two equations, arranged according to powers of x, be

$$ax^m+(by+cz)\,x^{m-1}+(dy^2+eyz+fz^2)\,x^{m-2}+\&c.=0,$$

$$a'x^n+(b'y+c'z)\,x^{n-1}+(d'y^2+e'yz+f'z^2)\,x^{n-2}+\&c.=0.$$

If now we eliminate x between these equations, since the coefficient of x^{m-1} is a homogeneous function of y and z of the first degree, that of x^{m-2} is a similar function of the second degree, and so on,—it follows from the last Article that the eliminant will be a homogeneous function of y and z of the mn^{th} degree. It follows then that mn values of y and z† can be found which will make the eliminant $=0$. If we substitute any one of these in the given equations, they will now have a common root when

* These equations may be considered as representing two curves of the m^{th} and n^{th} degrees respectively; the geometrical interpretation of the proposition of this Article being, that two such curves intersect in mn points. The equations are reduced to ordinary Cartesian equations by making $z=1$.

† The reader will remember that when we use homogeneous equations, the *ratio* of the variables is all with which we are concerned. Thus here, z may be taken arbitrarily, the corresponding value of y being determined by the equation in $y:z$.

solved for x (since their eliminant vanishes); and this value of x, combined with the values of y and z already found, gives one system of values satisfying the given equations. So we plainly have in all mn such systems of values. We shall, in Lesson X., give a method by which, when two equations have a common root, that common root can immediately be found.

Ex. To find the coordinates of the four points of intersection of the two conics $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, $a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0$. Arrange the equations according to the powers of x, and eliminate that variable, as in Art. 67; then the result is

$$\{(ab')\,y^2 + 2\,(af'')\,yz + (ac')\,z^2\}^2$$
$$+ 4\,[(ah')\,y + (ag')\,z]\,[(bh')\,y^3 + \{(bg') + 2\,(fh')\}\,y^2z + \{(ch') + 2\,(fg')\}\,z^2y + (cg')\,z^3] = 0,$$

where, as in Lesson I., we have written (ab') for $ab' - a'b$. This equation, solved for $y : z$, determines the values corresponding to the four points of intersection. Having found these, by substituting any one of them in both equations, and finding their common root, we obtain the corresponding value of $x : z$. We might have at once got the four values of $x : z$ by eliminating y between the equations, but substitution in the equations is necessary in order to find which value of y corresponds to each value of x. By making $z = 1$, what has been said is translated into the language of ordinary Cartesian coordinates.

74. *Any symmetric functions of the mn values which simultaneously satisfy the two equations can be expressed in terms of the coefficients of those equations.*

In order to be more easily understood, we first consider non-homogeneous equations in two variables. Then it is plain enough that we can so express symmetric functions involving either variable alone. For, eliminating y, we have an equation in x, in terms of whose coefficients can be expressed all symmetric functions of the mn values of x which satisfy both equations. Similarly for y. Thus, for example, in the case of two conics, x_1y_1 &c. being the coordinates of their points of intersection, we see at once how to express such symmetric functions as

$$x_1 + x_{11} + x_{111} + x_{1v}, \quad y^2_1 + y^2_{11} + y^2_{111} + y^2_{1v}, \text{ &c.},$$

and the only thing requiring explanation is how to express symmetric functions into which both variables enter, such as

$$x_1y_1 + x_{11}y_{11} + x_{111}y_{111} + x_{1v}y_{1v}.$$

To do this, we introduce a new variable, $t = \lambda x + \mu y$, and by the help of this assumed equation eliminate both x and y from the given equations. Thus y is immediately eliminated by substi-

tuting in both its value derived from $t = \lambda x + \mu y$, and then we have two equations of the m^{th} and n^{th} degrees in x, the eliminant of which will be of the mn^{th} degree in t, and its roots will be obviously $\lambda x_{,} + \mu y_{,}$, $\lambda x_{,,} + \mu y_{,,}$, &c., where $x_{,} y_{,}$, $x_{,,} y_{,,}$ are the values of x and y common to the two equations. The coefficients of this equation in t will of course involve λ and μ. We next form the sum of the k^{th} powers of the roots of this equation in t, which must plainly be $= (\lambda x_{,} + \mu y_{,})^k + (\lambda x_{,,} + \mu y_{,,})^k +$ &c. The coefficient, then, of λ^k in this sum will be $\Sigma x_{,}^{\,k}$: the coefficient of $\lambda^{k-1}\mu$ gives us $\Sigma x_{,}^{\,k-1} y_{,}$, and so on.

Little need be said in order to translate the above into the language of homogeneous equations. We see at once how to form symmetric functions involving two variables only, such as $\Sigma y_{,} z_{,,} z_{,,,} z_{,,,,}$, for these are found, as explained, Art. 61, from the homogeneous equation obtained on eliminating the remaining variable; the only thing requiring explanation is how to form symmetric functions involving all these variables, and this is done precisely as above, by substituting $t = \lambda x + \mu y$.

Ex. To form the symmetric functions of the coordinates of the four points common to two conics. The equation in the last Example gives at once

$$y_{,} y_{,,} y_{,,,} y_{,,,,} = (ac')^2 + 4\,(ag')\,(cg')\,; \quad z_{,} z_{,,} z_{,,,} z_{,,,,} = (ab')^2 + 4\,(ah')\,(bh')\,;$$

and, from symmetry, $x_{,} x_{,,} x_{,,,} x_{,,,,} = (bc')^2 + 4\,(bf')\,(cf')$,

$$-\Sigma\,(y_{,} y_{,,} y_{,,,} z_{,,,,}) = 4\,\{(ac')\,(af') + (ah')\,(cg') + (ag')\,(ch') + 2\,(ag')\,(fg')\},\ \&c.$$

To take an example of a function involving three variables, let us form

$$\Sigma\,(x_{,} y_{,} z^2_{,,} z^2_{,,,} z^2_{,,,,}),$$

which corresponds to $\Sigma\,(x'y')$ when the equations are written in the non-homogeneous form.

By the preceding theory we are to eliminate between the given equations, and $t = \lambda x + \mu y$; and the required function will be half the coefficient of $\lambda\mu$ in $\Sigma\,(t^2 z^2_{,} z^2_{,,} z^2_{,,,} z^2_{,,,,})$. If the result of elimination be

$$At^4 + (B\lambda + C\mu)\,t^3 z + (D\lambda^2 + E\lambda\mu + F\mu^2)\,t^2 z^2 + \&c.,$$

$$\Sigma\,(t^2 z^2_{,} z^2_{,,} z^2_{,,,} z^2_{,,,,}) = (B\lambda + C\mu)^2 - 2A\,(D\lambda^2 + E\lambda\mu + F\mu^2),$$

and $$\Sigma\,(x_{,} y_{,} z^2_{,,} z^2_{,,,} z^2_{,,,,}) = BC - AE.$$

By actual elimination

$$A = (ab')^2 + 4\,(ah')\,(bh'),\quad B = 4\,\{(ba')\,(bg') + (bf')\,(ah') + (bh')\,(af') + 2\,(bh')\,(gh')\},$$

$$C = 4\,\{(ab')\,(af') + (ag')\,(bh') + (ah')\,(bg') + 2\,(ah')\,(fh')\}$$

$$E = 4\,\{(ac')\,(bh') + (bc')\,(ah') - 2\,(af')\,(hf') - 2\,(bg')\,(hg') + 4\,(hf')\,(hg')\}.$$

75. *To form the eliminant of three homogeneous equations in three variables of the m^{th}, n^{th}, and p^{th} degrees respectively.*

The vanishing of the eliminant is the condition that a system

of values of x, y, z can be found to satisfy all three equations.* When this is the case, if we solve from any two of the equations, and substitute successively in the remaining one the values so found for x, y, z, some one of these sets of values must satisfy that equation, and therefore the product of all the results of substitution must vanish. Let x', y', z'; x'', y'', z'', &c. be the systems of values which satisfy the last two equations, which (Art. 73) are np in number: substitute these values in the first, and multiply together the np results $\phi(x', y', z')$, $\phi(x'', y'', z'')$, &c. The product will plainly involve only symmetric functions of x', y', z', &c., which (Art. 74) can all be expressed in terms of the coefficients of the last two equations; and, when they are so expressed, it is the eliminant required.

76. *The eliminant is a homogeneous function of the np^{th} order in the coefficients of the first equation; of the mp^{th} in those of the second; and of the mn^{th} in those of the third.*

For, each of the np factors $\phi(x', y', z')$ is a homogeneous function of the first degree in the coefficients of the first equation; and the expression of the symmetric functions in terms of the coefficients only involves coefficients of the last two equations, from solving which x', y', z', &c. were obtained. The eliminant is therefore of the np^{th} degree in the coefficients of the first equation; and in like manner its degree in the coefficients of the others may be inferred.

77. *The weight of the eliminant will be mnp;* that is to say, *If all the coefficients in the equations which multiply the first power of one of the variables, z, be affected with a suffix* 1, *those which multiply z^2 with a suffix* 2, *and so on; the sum of all the suffixes in each term of the eliminant will be equal to mnp.* In other words: *If all the coefficients which multiply z contain a new variable in the first degree;—if those which multiply z^2 contain it in the second and lower degrees, and so on; then the eliminant will contain this variable in the degree mnp.*

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* If the three equations represent curves, the vanishing of the eliminant is the condition that all three curves shall pass through a common point.

This is proved as in Art. 71. In the first place, it is evident that if a homogeneous equation of the m^{th} degree be satisfied by values x', y', z'; and if the equation be altered by multiplying each coefficient by a power of λ, equal to the power of z, which the coefficient multiplies, then the equation so transformed will be satisfied by the values $\lambda x', \lambda y', z'$; or, in general, that the result of substituting $\lambda x', \lambda y', z'$ in the transformed equation is λ^m times the result of substituting x', y', z' in the untransformed. Thus, take the equation $x^3+y^3-z^3-z^2x-zy^2$, the transformed is $x^3+y^3-\lambda^3z^3-\lambda^2z^2x-\lambda zy^2$; and, obviously, the result of substituting $\lambda x', \lambda y', z'$ in the second is λ^3 times the result of substituting x', y', z' in the given equation. If, then, the three given equations be all transformed by multiplying each coefficient by a power of λ equal to the power of z, which the coefficient multiplies, it follows, if x', y', z' be one of the systems of values which satisfy the two last of the original equations, that the transformed equations will be satisfied by $(\lambda x', \lambda y', z')$, and the result of substituting these values in the first will be $\lambda^m\phi(x', y', z')$. The eliminant, then, which is the product of np factors of the form $\phi(x', y', z')$ will be multiplied by λ^{mnp}. If, then, any term in the eliminant be $a_kb_lc_m$, &c., where the suffix corresponds to the power of z, which the coefficient multiplies, since the alteration of a_k into $\lambda^k a_k$, b_l into $\lambda^l b_l$, &c., multiplies the term by λ^{mnp}, we must have $k+l+\&c.=mnp$. Q.E.D.

78. It is proved, in like manner, that three equations are in general satisfied by mnp common values; that any symmetric function of these values can be expressed in terms of their coefficients; and that we can form the eliminant of four equations by solving from any three of them, substituting successively in the fourth each of the systems of values so found, forming the product of the results of substitution, and then, by the method of symmetric functions, expressing the product in terms of the coefficients of the equations. In this way we can form the eliminant of any number of equations; and we have the following general theorems: *The eliminant of k equations in $k-1$ independent variables is a homogeneous function of the coefficients*

of each equation, whose order is equal to the product of the degrees of all the remaining equations. If each coefficient in all the equations be affected with a suffix equal to the power of any one variable which it multiplies, then the sum of the suffixes in every term of the eliminant will be equal to the product of the degrees of all the equations. And, again, *if we are given k equations in k variables, the number of systems of common values of the variables, which can be found to satisfy all the equations, will be equal to the product of the orders of the equations.*

LESSON IX.

EXPRESSION OF ELIMINANTS AS DETERMINANTS.

79. THE method of elimination by symmetric functions is, in a theoretical point of view, perhaps preferable to any other, it being universally applicable to equations in any number of variables; yet as (in the absence of tables of symmetric functions) it is not very expeditious in practice, and does not yield its results in the most convenient form, we shall in this Lesson give an account of some other methods of elimination. The following is the method which most obviously presents itself. It is in substance identical with what is called elimination by the process of finding the greatest common measure. We have already seen that the eliminant of two linear equations $ax+b=0$, $a'x+b'=0$ is the determinant $ab'-ba'=0$. If now we have two quadratic equations

$$ax^2+bx+c=0,\quad a'x^2+b'x+c'=0,$$

multiplying the first by a', the second by a, and subtracting, we get

$$(ab')\,x+(ac')=0;$$

and, again, multiplying the first by c', the second by c, subtracting, and dividing by x, we get

$$(ac')\,x+(bc')=0.$$

The problem is now reduced to elimination between two linear equations, and the result is

$$(ac')^2 + (ba')(bc') = 0.$$

80. So, again, if we have two cubic equations

$$ax^3 + bx^2 + cx + d = 0, \quad a'x^3 + b'x^2 + c'x + d' = 0,$$

we multiply the first by a', the second by a, and subtract; and also multiply the first by d', the second by d, subtract and divide by x. The problem is thus reduced to elimination between the two quadratics

$$(ab')x^2 + (ac')x + (ad') = 0, \quad (ad')x^2 + (bd')x + (cd') = 0.$$

By the last article the result is

$$\{(ad')^2 - (ab')(cd')\}^2 + \{(ad')(ac') - (ab')(bd')\}\{(ad')(db') - (ac')(dc')\} = 0.$$

Now it is to be observed that the equation

$$(ab')(cd') + (ac')(db') + (ad')(bc') = 0$$

is identically true. Consequently when we multiply out, the preceding result becomes divisible by (ad'), and the reduced result is

$$(ad')^3 - 2(ad')(ab')(cd') - (ad')(ac')(bd') + (ac')^2(cd') + (bd')^2(ab') - (ab')(bc')(cd') = 0.$$

The reason that in this process the irrelevant factor (ad') is introduced is that, if $ad' = a'd$, and therefore a to a' in the same ratio as d to d', we must get results differing only by a factor, if from the first equation multiplied by a' we subtract the second equation multiplied by a, or, if from the first equation multiplied by d', we subtract the second equation multiplied by d. Thus, on the supposition $(ad') = 0$, even though the original two cubics have not a common factor, the two quadratics to which we reduce them would have a common factor In general then, when we eliminate by this process, irrelevant factors are introduced, and therefore other methods are preferable.

81. *Euler's Method.* If two equations of the m^{th} and n^{th} degrees respectively have a common factor of the first degree, we must obtain identical results, whether we multiply the first

equation by the remaining $n-1$ factors of the second, or the second by the remaining $m-1$ factors of the first. If then we multiply the first by an arbitrary function of the $(n-1)^{th}$ degree, which, of course, introduces n arbitrary constants; if we multiply the second by an arbitrary function of the $(m-1)^{th}$ degree, introducing thus m constants; and if we then equate, term by term, the two equations of the $(m+n-1)^{th}$ degree so formed, we shall have $m+n$ equations, from which we can eliminate the $m+n$ introduced constants, which all enter into those equations only in the first degree; and we thus obtain, in the form of a determinant, the eliminant of the two given equations.

Ex. To eliminate between $ax^2+bxy+cy^2=0$, $a'x^2+b'xy+c'y^2=0$.

We are to equate, term by term,

$$(Ax+By)(ax^2+bxy+cy^2) \text{ and } (A'x+B'y)(a'x^2+b'xy+c'y^2).$$

The four resulting equations are

$$\begin{aligned} Aa \ldots\ldots - A'a' \ldots\ldots &= 0, \\ Ab + Ba - A'b' - B'a' &= 0, \\ Ac + Bb - A'c' - B'b' &= 0, \\ Bc \ldots\ldots - B'c' &= 0, \end{aligned}$$

from which, eliminating A, B, A', B', the result is the determinant

$$\begin{vmatrix} a, & 0, & a', & 0 \\ b, & a, & b', & a' \\ c, & b, & c', & b' \\ 0, & c, & 0, & c' \end{vmatrix}.$$

82. This method may be extended to find the conditions that the equations should have two common factors. In this case it is evident, in like manner, that we shall obtain the same result whether we multiply the first by the remaining $n-2$ factors of the second, or the second by the remaining $m-2$ factors of the first. As before, then, we multiply the first by an arbitrary function of the $n-2$ degree (introducing $n-1$ constants), and the second by an arbitrary function of the $m-2$ degree; and equating, term by term, the two equations of the $m+n-2$ degree so found, we have $m+n-1$ equations, from any $m+n-2$ of which, eliminating the $m+n-2$ introduced constants, we obtain $m+n-1$ conditions, equivalent, of course, to only two independent conditions.

Ex. To find the conditions that

$$ax^3 + bx^2y + cxy^2 + dy^3 = 0, \quad a'x^3 + b'x^2y + c'xy^2 + d'y^3 = 0,$$

should have two common factors. Equating

$$(Ax + By)(ax^3 + bx^2y + cxy^2 + dy^3) = (A'x + B'y)(a'x^3 + b'x^2y + c'xy^2 + d'y^3),$$

we have

$$\begin{aligned} Aa \ldots\ldots - A'a' \ldots\ldots &= 0, \\ Ab + Ba - A'b' - B'a' &= 0, \\ Ac + Bb - A'c' - B'b' &= 0, \\ Ad + Bc - A'd' - B'c' &= 0, \\ Bd \ldots\ldots - B'd' &= 0, \end{aligned}$$

from which, eliminating A, B, A', B', we have the system of determinants [for the notation used, see Art. 3],

$$\begin{Vmatrix} a, & b, & c, & d, & 0 \\ 0, & a, & b, & c, & d \\ a', & b', & c', & d', & 0 \\ 0, & a', & b', & c', & d' \end{Vmatrix} = 0.$$

83. *Sylvester's dialytic method.* This method is identical in its results with Euler's, but simpler in its application, and more easily capable of being extended. Multiply the equation of the m^{th} degree by x^{n-1}, $x^{n-2}y$, $x^{n-3}y^2$, &c.; and the second equation by x^{m-1}, $x^{m-2}y$, $x^{m-3}y^2$, &c., and we thus get $m+n$ equations, from which we can eliminate linearly the $m+n$ quantities x^{m+n-1}, $x^{m+n-2}y$, $x^{m+n-3}y^2$, &c., considered as independent unknowns. Thus, in the case of two quadratics, multiply both by x and by y, and we get the equations

$$\begin{aligned} ax^3 + bx^2y + cxy^2 &= 0, \\ ax^2y + bxy^2 + cy^3 &= 0, \\ a'x^3 + b'x^2y + c'xy^2 &= 0, \\ a'x^2y + b'xy^2 + c'y^3 &= 0, \end{aligned}$$

from which, eliminating x^3, x^2y, xy^2, y^3, we get the same determinant as before,

$$\begin{vmatrix} a, & b, & c, & \\ & a, & b, & c \\ a', & b', & c', & \\ & a', & b', & c' \end{vmatrix}.$$

In general, it is evident by this method, that the eliminant is expressed as a determinant of which n rows contain the coefficients of the first equation, and m rows contain the coefficients

of the second. Thus we obtain the rule already stated for the order of the eliminant in the coefficients of each equation.

Ex. 1. Eliminate x between $a + bx + cx^2 = 0$, $x^3 = 1$.
Multiply the former by x, and put $x^3 = 1$, and we get

$$c + ax + bx^2 = 0;$$

from this similarly

$$b + cx + ax^2 = 0,$$

and the eliminant is in determinant form and expanded

$$\begin{vmatrix} a, & b, & c \\ c, & a, & b \\ b, & c, & a \end{vmatrix} = - \begin{vmatrix} a, & b, & c \\ b, & c, & a \\ c, & a, & b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

It can also be found by the method of symmetric functions: let ω be an imaginary cube root of unity, then the roots of the second equation substituted in the first give the eliminant as

$$(a + b + c)\,(a + b\omega + c\omega^2)\,(a + b\omega^2 + c\omega),$$

which is thus the value of the determinant.

Ex. 2. Similarly eliminating between $a + bx + cx^2 + dx^3 = 0$, $x^4 = 1$,

$$\begin{vmatrix} a, & b, & c, & d \\ d, & a, & b, & c \\ c, & d, & a, & b \\ b, & c, & d, & a \end{vmatrix} = - \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & a \\ c, & d, & a, & b \\ d, & a, & b, & c \end{vmatrix} = \{(a + c)^2 - (b + d)^2\}\,\{(a - c)^2 + (b - d)^2\}$$

$$= (a + b + c + d)\,(a - b + c - d)\,(a + bi - c - di)\,(a - bi - c + di).$$

Ex. 3. Generally, in a similar manner for ω an imaginary n^{th} root of unity, the determinant

$$\begin{vmatrix} a, & b, & c, & \ldots l \\ l, & a, & b, & \ldots k \\ k, & l, & a, & \ldots\ldots \\ \ldots\ldots & & & \\ b, & c, & d, & \ldots a \end{vmatrix} = \begin{array}{l} (a + \ b \ + \ c \ + \ldots + l) \\ (a + \ b\omega \ + c\omega^2 + \ldots + l\omega^{n-1}) \\ (a + \ b\omega^2 \ + \ldots\ldots\ldots + l\omega^{n-2}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (a + b\omega^{n-1} + \ldots\ldots\ldots + l\omega). \end{array}$$

Ex. 4. Various determinants may be found by this method of elimination. Starting from the equations $a_1x + a_2y = 0$, $b_1x + b_2y = 0$, they are consistent if $(a_1b_2) = 0$. Take the squares and product of these expressions, they also vanish together if the original equations be consistent; hence, eliminating x^2, xy, y^2 dialytically,

$$\begin{vmatrix} a_1^2, & 2a_1a_2, & a_2^2 \\ a_1b_1, & a_1b_2 + a_2b_1, & a_2b_2 \\ b_1^2, & 2b_1b_2, & b_2^2 \end{vmatrix}$$

must be a power of (a_1b_2), and it is obviously $(a_1b_2)^3$.

Similarly $(a_1b_2)^6$ can be written down as a determinant of order 4. Again,

$$\begin{aligned} (b_1x + b_2y)\,(c_1x + c_2y) &= 0, \\ (c_1x + c_2y)\,(a_1x + a_2y) &= 0, \\ (a_1x + a_2y)\,(b_1x + b_2y) &= 0, \end{aligned}$$

are consistent if $(b_1c_2)\,(c_1a_2)\,(a_1b_2)$ vanish. Hence this product may be written

$$\begin{vmatrix} b_1c_1, & b_1c_2 + b_2c_1, & b_2c_2 \\ c_1a_1, & c_1a_2 + c_2a_1, & c_2a_2 \\ a_1b_1, & a_1b_2 + a_2b_1, & a_2b_2 \end{vmatrix}.$$

And in like manner all determinants of the matrix

$$\begin{vmatrix} a_1^2, & b_1^2, & c_1^2, & b_1c_1, & c_1a_1, & a_1b_1 \\ 2a_1a_2, & 2b_1b_2, & 2c_1c_2, & b_1c_2+b_2c_1, & c_1a_2+c_2a_1, & a_1b_2+a_2b_1 \\ a_2^2, & b_2^2, & c_2^2, & b_2c_2, & c_2a_2, & a_2b_2 \end{vmatrix}$$

are products of the third order of the quantities (b_1c_2), (c_1a_2), (a_1b_2).

84. *Bezout's method.* This process also expresses the eliminant in the form of a determinant, but one which can be more rapidly calculated than the preceding. The general method will, perhaps, be better understood if we apply it first to the particular case of the two equations of the fourth degree

$$ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=0, \quad a'x^4+b'x^3y+c'x^2y^2+d'xy^3+e'y^4=0.$$

Multiplying the first by a', the second by a, and subtracting, the first term in each is eliminated, and the result, being divisible by y, gives

$$(ab')\,x^3+(ac')\,x^2y+(ad')\,xy^2+(ae')\,y^3=0.$$

Again, multiply the first by $a'x+b'y$, and the second by $ax+by$, and the two first terms in each are eliminated, and the result, being divided by y^2, gives

$$(ac')\,x^3+\{(ad')+(bc')\}\,x^2y+\{(ae')+(bd')\}\,xy^2+(be')\,y^3=0.$$

Next, multiply the first by $a'x^2+b'xy+c'y^2$; and the second by $ax^2+bxy+cy^2$; subtract, and divide by y^3; when we get

$$(ad')\,x^3+\{(ae')+(bd')\}\,x^2y+\{(be')+(cd')\}\,xy^2+(ce')\,y^3=0.$$

Lastly, multiply the first by $a'x^3+b'x^2y+c'xy^2+d'y^3$; the second by $ax^3+bx^2y+cxy^2+dy^3$; subtract, and divide by y^4; when we get

$$(ae')\,x^3+(be')\,x^2y+(ce')\,xy^2+(de')\,y^3=0.$$

From the four equations thus formed, we can eliminate linearly the four quantities x^3, x^2y, xy^2, y^3, and obtain for our result the determinant

$$\begin{vmatrix} (ab'), & (ac') & , & (ad') & , & (ae') \\ (ac'), & (ad')+(bc'), & (ae')+(bd') & , & (be') \\ (ad'), & (ae')+(bd'), & (be')+(cd'), & (ce') \\ (ae'), & (be'), & (ce'), & (de') \end{vmatrix}.$$

85. The process here employed is so evidently applicable to any two equations, both of the n^{th} degree, that it is unnecessary to make a formal statement of the general proof. On inspection of the determinant obtained in the last article, the law of its formation is apparent, and we can at once write down the determinant which is the eliminant between two equations of the fifth degree by simply continuing the series of terms, writing an (af') after every (ae'), &c. Thus the eliminant is

$$\begin{vmatrix} (ab'), & (ac') & , (ad') & , (ae') & , (af') \\ (ac'), & (ad')+(bc'), & (ae')+(bd') & , (af')+(be') & , (bf') \\ (ad'), & (ae')+(bd'), & (af')+(be')+(cd'), & (bf')+(ce') & , (cf') \\ (ae'), & (af')+(be'), & (bf')+(ce'), & (cf')+(de'), & (df') \\ (af'), & (bf'), & (cf'), & (df'), & (ef') \end{vmatrix}$$

It appears hence that in the eliminant every term must contain a or a'; as was evident beforehand, since if both of these were $= 0$, the equations would evidently have the common factor $y = 0$.

It appears also that those terms which contain a or a' only in the first degree are (ab') multiplied by the eliminant of the equations got by making a and $a' = 0$ in the given equations. For every element in the determinant written above must contain a constituent from the first row, and also one from the first column; but as all the constituents of the first row or column contain a or a', the only terms which contain a and a' in only the first degree are (ab') multiplied by the corresponding minor; and this, when a and a' are made $= 0$, is the next lower eliminant.

86. It only remains to shew that the process here employed is applicable when the equations are of different dimensions; and, as before, we commence with the following example:

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, \quad a'x^2 + b'xy + c'y^2 = 0.$$

Multiply the first equation by a', the second by ax^2, and subtract, when we have

$$(ba')\,x^3 + (ca')\,x^2y + (da')\,xy^2 + (ea')\,y^3 = 0.$$

In like manner, multiply the first by $a'x + b'y$, and the second by $(ax + by)\,x^2$, and we get

$$(ca')\,x^3 + \{(cb') + (da')\}\,x^2y + \{(db') + (ca')\}\,xy^2 + (eb')\,y^3 = 0.$$

This process can be carried no further; but if we join to the two equations just obtained the two equations got by multiplying the second of the original equations by x and by y, we have four equations from which to eliminate x^3, x^2y, xy^2, y^3.

And in general, when the degrees of the equations are unequal, m being the greater, it will be found that the process of Art. 84 gives us n equations of the $(m-1)^{th}$ degree, each of these equations being of the first order in the coefficients of *each* equation: to which we are to add the $m-n$ equations found by multiplying the second equation by x^{m-n-1}, $x^{m-n-2}y$, &c., and we can then eliminate the m quantities x^{m-1}, $x^{m-2}y$, &c., from the m equations we have formed. Every row of the determinant contains the coefficients of the second equation, but only n rows contain the coefficients of the first. The eliminant is, therefore, as it ought to be, of the n^{th} degree in the coefficients of the first, and of the m^{th} in those of the second equation.

87. *Cayley's statement of Bezout's method.* If two equations $\phi(x, y)$, $\psi(x, y)$ have a common root, then it must be possible to satisfy any equation of the form $\phi + \lambda\psi = 0$, independently of any particular value of λ. Take, then, the equation

$$\phi(x, y)\,\psi(x', y') - \phi(x', y')\,\psi(x, y) = 0;$$

which, if ϕ and ψ have a common factor, can be satisfied independently of any particular values of x' and y'. We may in the first place divide it by $xy' - yx'$, which is obviously a factor: next equate to 0 the coefficients of the several powers of x', y'; and then eliminate the powers of x and y as if they were independent variables, when the result comes out in precisely the same form as by the method of Art. 84.

Ex. To eliminate between $ax^2 + bxy + cy^2 = 0$, $a'x^2 + b'xy + c'y^2 = 0$.

$(ax^2 + bxy + cy^2)(a'x'^2 + b'x'y' + c'y'^2) - (a'x^2 + b'xy + c'y^2)(ax'^2 + bx'y' + cy'^2)$, when divided by $xy' - yx'$ gives

$$\{(ab')\,x + (ac')\,y\}\,x' + \{(ac')\,x + (bc')\,y\}\,y' = 0;$$

and, eliminating x, y between the coefficients of x' and y', separately equated to 0, we get the eliminant

$$(ac')^2 + (ba')(bc') = 0.$$

88. We proceed now to the theory of functions of three variables, the eliminant of which, however, except in particular cases, has not been expressed as a determinant, though it can

always be expressed as the quotient of one determinant divided by another. We shall shew, in the first place, how to form a function of great importance in the theory of elimination. Given k equations in k variables, $u=0$, $v=0$, $w=0$, &c., if we write u_1, u_2, u_3, &c. for $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, &c., then the determinant

$$\begin{vmatrix} u_1, & u_2, & u_3, & \&c. \\ v_1, & v_2, & v_3, & \&c. \\ w_1, & w_2, & w_3, & \&c. \\ \&c. & & & \end{vmatrix}$$

is called Jacobi's determinant, or simply the *Jacobian* of the given equations, and will be denoted in what follows by the letter J.

89. *If any number of equations are satisfied by a common system of values, that system will satisfy the Jacobian; and when the equations are of the same degree, it will also satisfy the derived equations of the Jacobian with regard to each of the variables.*

The proof of this for three variables applies in general. By the theorem of homogeneous functions we have

$$xu_1 + yu_2 + zu_3 = nu,$$
$$xv_1 + yv_2 + zv_3 = n'v,$$
$$xw_1 + yw_2 + zw_3 = n''w.$$

Now if, as in Lesson IV., we write the minors of the Jacobian, obtained by suppressing the row and column containing u_1, v_1, &c., U_1, V_1, &c., then if we solve these equations, we find (Art. 29) $Jx = U_1 nu + V_1 n'v + W_1 n''w$, from which it appears at once that if u, v, w vanish, J will vanish too. Again, differentiating the equation just found, we have, for $n' = n'' = n$,

$$J + x\frac{dJ}{dx} = nu\frac{dU_1}{dx} + nv\frac{dV_1}{dx} + nw\frac{dW_1}{dx} + n(u_1U_1 + v_1V_1 + w_1W_1),$$

$$x\frac{dJ}{dy} = nu\frac{dU_1}{dy} + nv\frac{dV_1}{dy} + nw\frac{dW_1}{dy} + n(u_2U_1 + v_2V_1 + w_2W_1).$$

But remembering (Art. 27) that

$$u_1U_1 + v_1V_1 + w_1W_1 = J, \quad u_2U_1 + v_2V_1 + w_2W_1 = 0,$$

we see that the supposition $u=0$, $v=0$, $w=0$ (in consequence of which J is also $=0$) makes $\frac{dJ}{dx}$, $\frac{dJ}{dy}$ also to vanish.

90. We can now express as a determinant the eliminant of three equations, each of the second degree. For their Jacobian is of the third degree, and therefore its differentials are of the second. We have thus three new equations of the second degree, which will be also satisfied by any system of values common to the given equations. From the six equations, then, $u, v, w, \frac{dJ}{dx}, \frac{dJ}{dy}, \frac{dJ}{dz}$, we can eliminate the six quantities $x^2, y^2, z^2, yx, zx, xy$, and so form the determinant required.

Again, if the equations are all of the third degree, J is of the sixth, and its differentials of the fifth, and if we multiply each of the three given equations by $x^2, y^2, z^2, yz, zx, xy$, we obtain eighteen equations, which, combined with the three differentials of the Jacobian, enable us to eliminate dialytically the twenty-one quantities, x^5, x^4y, &c., which enter into an equation of the fifth degree. This process, however, cannot, without modification, be extended further.

Ex. 1. The Jacobian of two homogeneous equations may likewise be employed to write down in determinant form their eliminants. Thus, for

$$ax^2 + 2bxy + cy^2 = 0,$$

$$a'x^2 + 2b'xy + c'y^2 = 0,$$

writing the Jacobian $a_0x^2 + 2a_1xy + a_2y^2$,

the eliminant is $\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a_0, & a_1, & a_2 \end{vmatrix}$.

Ex. 2. For two cubics $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$,

$$a'x^3 + 3b'x^2y + 3c'xy^2 + d'y^3 = 0,$$

the Jacobian may be written

$$a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4,$$

and since its two derived functions vanish for a common value, the eliminant is

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \end{vmatrix}.$$

Similarly for two biquadratics with a like notation, the eliminant is

$$\begin{vmatrix} a, & b, & c, & d, & e, & 0 \\ a', & b', & c', & d', & e', & 0 \\ 0, & a, & b, & c, & d, & e \\ 0, & a' & b', & c', & d', & e' \\ a_0, & a_1, & a_2, & a_3, & a_4, & a_5 \\ a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \end{vmatrix}.$$

Ex. 3. As in Ex. 4, Art. 83, we can easily determine the following and similar identities:

$$(a_1b_2c_3)^4 = \begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & 2a_2a_3, & 2a_3a_1, & 2a_1a_2 \\ b_1^2, & b_2^2, & b_3^2, & 2b_2b_3, & 2b_3b_1, & 2b_1b_2 \\ c_1^2, & c_2^2, & c_3^2, & 2c_2c_3, & 2c_3c_1, & 2c_1c_2 \\ b_1c_1, & b_2c_2, & b_3c_3, & b_2c_3+b_3c_2, & b_3c_1+b_1c_3, & b_1c_2+b_2c_1 \\ c_1a_1, & c_2a_2, & c_3a_3, & c_2a_3+c_3a_2, & c_3a_1+c_1a_3, & c_1a_2+c_2a_1 \\ a_1b_1, & a_2b_2, & a_3b_3, & a_2b_3+a_3b_2, & a_3b_1+a_1b_3, & a_1b_2+a_2b_1 \end{vmatrix},$$

$$(b_1c_2d_3)(c_1d_2a_3)(d_1a_2b_3)(a_1b_2c_3) = \begin{vmatrix} b_1c_1, & b_2c_2, & b_3c_3, & b_2c_3+b_3c_2, & b_3c_1+b_1c_3, & b_1c_2+b_2c_1 \\ c_1a_1, & c_2a_2, & c_3a_3, & c_2a_3+c_3a_2, & c_3a_1+c_1a_3 & c_1a_2+c_2a_1 \\ a_1b_1, & a_2b_2, & a_3b_3, & a_2b_3+a_3b_2, & a_3b_1+a_1b_3, & a_1b_2+a_2b_1 \\ a_1d_1, & a_2d_2, & a_3d_3, & a_2d_3+a_3d_2, & a_3d_1+a_1d_3, & a_1d_2+a_2d_1 \\ b_1d_1, & b_2d_2, & b_3d_3, & b_2d_3+b_3d_2, & b_3d_1+b_1d_3, & b_1d_2+b_2d_1 \\ c_1d_1, & c_2d_2, & c_3d_3, & c_2d_3+c_3d_2, & c_3d_1+c_1d_3, & c_1d_2+c_2d_1 \end{vmatrix}.$$

91. Dr. Sylvester has shewn that the eliminant can always be expressed as a determinant when the three equations are of the same degree. Let us take, for an example, three equations of the fourth degree. Multiply each by the six terms (x^2, xy, y^2, &c.) of an equation of the second degree [or generally by the $\frac{1}{2}n(n-1)$ terms of an equation of the degree $(n-2)$]. We thus form eighteen $[\frac{3}{2}n(n-1)]$ equations. But since these equations, being now of the sixth $[2n-2]$ degree, consist of twenty-eight $[n(2n-1)]$ terms, we require ten $[\frac{1}{2}n(n+1)]$ additional equations to enable us to eliminate dialytically all the powers of the variables. These equations are formed as follows: The first of the three given equations can be written in the form $Ax^4+By+Cz$, the second and third in the form

$$A'x^4+B'y+C'z, \quad A''x^4+B''y+C''z;$$

and the determinant $(AB'C'')$ which is of the sixth degree in the variables must obviously vanish for any values which satisfy all the given equations. We should form two similar determinants by decomposing the equations into the form $Ay^4+Bx+Cz$, $Az^4+Bx+Cy$. So again we might decompose

the equations into the forms $Ax^3 + By^2 + Cz$, $A'x^3 + B'y^2 + C'z$, $A''x^3 + B''y^2 + C''z$) for every term not divisible by x^3 or y^2 must be divisible by z); and then we obtain another determinant $(AB'C'')$ which will vanish when the equations vanish together. There are six determinants of this form got by interchanging x, y, and z in the rule for decomposing the equations. Lastly, decomposing into the form $Ax^2 + By^2 + Cz^2$, &c., we get a single determinant, which, added to the nine equations already found, makes the ten required. In general, we decompose the equations into the form $Ax^\alpha + By^\beta + Cz^\gamma$, such that $\alpha + \beta + \gamma = n + 2$, and form the determinant $(AB'C'')$; and it can be very easily proved that the number of integer solutions of the equation $\alpha + \beta + \gamma = n + 2$ is $\frac{1}{2}n(n+1)$, exactly the number required.

92. When the degrees of the equations are different, it is not possible to form in this way a determinant, which shall give the eliminant clear of extraneous factors. The reason why such factors are introduced, and the method by which they are to be got rid of, will be understood from the following theory, due to Prof. Cayley: Let us take for simplicity three equations, u, v, w, all of the second degree. If we attempt to eliminate dialytically by multiplying each by x, y, z, we get nine equations, which are not sufficient to eliminate the ten quantities x^3, x^2y, &c. Again, if we multiply each equation by the six quantities, x^2, xy, y^2, &c., we have eighteen equations, which are *more than* sufficient to eliminate the fifteen quantities x^4, x^3y, &c. If we take at pleasure any fifteen of these equations, and form their determinant, we shall indeed have the eliminant, but it will be multiplied by an extraneous factor, since the determinant is of the fifteenth degree in the coefficients, while the eliminant is only of the twelfth (Art. 76, $mn + np + pm = 12$, when $m = n = p = 2$). The reason of this is, that the eighteen equations we have formed are not independent, but are connected by three linear relations. In fact, if we write the identity $uv = vu$, and then replace the first u by its value, $ax^2 + by^2 +$ &c., and in like manner with the v on the right-hand side of the equation, we get

$$ax^2v + by^2v + cz^2v + 2fyzv + \&c. = a'x^2u + b'y^2u + \&c.$$

In like manner, from the identities $vw = wv$, $wu = uw$, we get two other identical relations connecting the quantities x^2u, y^2u, x^2v, x^2w, &c. The question then comes to this: "If there be $m+p$ linear equations in m variables, but these equations connected by p linear relations so as to be equivalent only to m independent equations, how to express most simply the condition that all the equations can be made to vanish together." In the present case $m = 15$, $p = 3$.

93. Let us, for simplicity, take an example with numbers not quite so large, for instance, $m = 3$, $p = 1$. That is to say, let us consider four equations, s, t, u, v, where $s = a_1x + b_1y + c_1z$, $t = a_2x + b_2y + c_2z$, &c., these equations not being independent, but satisfying the relation, $D_1s + D_2t + D_3u + D_4v = 0$. Now I say, in the first place, that if we form the determinant $(a_1b_2c_3)$ of any three of these equations, s, t, u, this must contain D_4 as a factor. For if $D_4 = 0$, we shall have s, t, u connected by a linear relation, so that any values which satisfied both s and t should satisfy u also; and therefore the supposition $D_4 = 0$ would cause the determinant $(a_1b_2c_3)$ to vanish. And, in the second place, I say that we get the same result (or, at most, one differing only in sign) whether we divide $(a_1b_2c_3)$ by D_4 or $(a_1b_2c_4)$ by D_3. For (Art. 15) $D_4(a_1b_2c_4)$ is the same as the determinant of which the first row is a_1, b_1, c_1, the second, a_2, b_2, c_2, and the third, D_4a_4, D_4b_4, D_4c_4; but we may substitute for D_4a_4 its value $-D_1a_1 - D_2a_2 - D_3a_3$, and in like manner for D_4b_4, D_4c_4. The determinant would then (Art. 18) be resolvable into the sum of three others; but two of these would vanish, having two rows the same, and there would remain $D_4(a_1b_2c_4) = -D_3(a_1b_2c_3)$. It follows, then, that the eliminant of the system may be expressed in any of the equivalent forms obtained by forming the determinant $(a_1b_2c_3)$ of any three of the equations, and dividing by the remaining constant D_4.

Suppose now that we had five equations s, t, u, v, w, connected by two linear relations $D_1s + D_2t + D_3u + D_4v + D_5w = 0$, $E_1s + E_2t + E_3u + E_4v + E_5w = 0$. Eliminating w from these relations, we have $(D_1E_5)s + (D_2E_5)t + (D_3E_5)u + (D_4E_5)v = 0$, and we see, precisely as before, that the supposition $(D_4E_5) = 0$

would cause the determinant $(a_1 b_2 c_3)$ to vanish; and that we get the same result whether we divide $(a_1 b_2 c_3)$ by $(D_4 E_5)$ or divide the determinant of any other three of the equations by the complemental determinant answering to $(D_4 E_5)$. This reasoning may be extended to any number of equations connected by any number of relations, and we are led to the following general rule for finding the eliminant of the system in its simplest form. Write down the constants in the $m+p$ equations, and complete them into a square form by adding the constants in the p relations; thus

$$\begin{array}{llll|ll} s; & a_1, & b_1, & c_1 & D_1, & E_1 \\ t; & a_2, & b_2, & c_2 & D_2, & E_2 \\ u; & a_3, & b_3, & c_3 & D_3, & E_3 \\ v; & a_4, & b_4, & c_4 & D_4, & E_4 \\ w; & a_5, & b_5, & c_5 & D_5, & E_5, \end{array}$$

then the eliminant in its most reduced form is the determinant of any m rows of the left-hand or equation columns, divided by the determinant got by erasing these rows in the right-hand columns.

Thus, then, in the example of the last Article, we take the determinant of any fifteen of the equations, and, dividing it by a determinant formed with three of the relation rows, obtain the eliminant, which is of the twelfth degree, as it ought to be.

94. And, in general, given three equations of the m^{th}, n^{th}, and p^{th} degrees, we form a number of equations of the degree $m+n+p-2$, by multiplying the first equation by all the terms x^{m+p-2}, $x^{m+p-3}y$, and so on. We should in this manner have

$$\tfrac{1}{2}(n+p-1)(n+p)+\tfrac{1}{2}(p+m-1)(p+m)+\tfrac{1}{2}(m+n-1)(m+n)$$

equations. But the number of terms, $x^{m+n+p-2}$, &c., to be eliminated from the equations formed, is $\frac{1}{2}(m+n+p-1)(m+n+p)$, or, in general, less than the number of equations. But again, if we consider the identity $uv=vu$, which is of the degree $m+n$, and multiply it by the several terms x^{p-2}, &c., we get $\frac{1}{2}(p-1)p$ identical relations between the system of equations we have

formed; and in like manner $\frac{1}{2}(n-1)n+\frac{1}{2}(m-1)m$ other identities; and the number of identities subtracted from the number of equations leaves exactly the number of variables to be eliminated, and gives the eliminant in the right degree.

95. If we had four equations in four variables, we should proceed in like manner, and it would be found then that the case would arise of our having $m+n$ linear equations in m variables, these equations not being independent, but connected by $n+p$ relations; these latter relations again not being independent, but connected by p other relations. And in order to find the reduced eliminant of such a system, we should divide the determinant of any m of the equations by a quantity which is itself the quotient of two determinants. I think it needless to go into further details, but I thought it necessary to explain so much of the theory, the above being, as far as I know, the only general theory of the expression of eliminants as determinants; since whenever, in the application of the dialytic method, any of the equations is multiplied by terms exceeding its own degree, we shall be sure to have a number of equations greater than the number of quantities which we want to eliminate.

LESSON X.

DETERMINATION OF COMMON ROOTS.

96. When the eliminant of any number of equations vanishes, these equations can be satisfied by a common system of values, and we purpose in this Lesson to shew how that system of values can be found without actually solving the equations. The method is the same whatever be the number of the variables; but for greater simplicity we commence with the system of two equations, $\phi=0$, $\psi=0$, where

$$\phi = a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \&c. = 0,$$
$$\psi = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \&c. = 0.$$

Let us suppose that some root of the second equation, $x=\alpha$ satisfies the first, and therefore that R the eliminant of the system vanishes. Now in ϕ we may alter the coefficients (a_m into a_m+A_m, a_{m-1} into $a_{m-1}+A_{m-1}$, &c.); and the transformed equation

$$a_m x^m + a_{m-1}x^{m-1} + \&c. + A_m x^m + A_{m-1}x^{m-1} + \&c. = 0$$

will obviously still be satisfied by the value $x=\alpha$, provided only that the increments A_m, A_{m-1}, &c. are connected by the single relation

$$A_m\alpha^m + A_{m-1}\alpha^{m-1} + \&c. = 0,$$

since the remaining part of the equation, by hypothesis, vanishes for $x=\alpha$. The transformed equation then has a root common with ψ, and therefore the eliminant between μ and that transformed equation vanishes. But this eliminant is obtained from R, the eliminant of ϕ and ψ, by altering in it a_m into a_m+A_m, &c. The eliminant so transformed is

$$R + \left\{A_m\frac{dR}{da_m} + A_{m-1}\frac{dR}{da_{m-1}} + \&c.\right\} + \&c. = 0.$$

We have $R=0$ by hypothesis; and since the increments A_m, &c. may be as small as we please, the terms containing the first powers of these increments must vanish separately. We have then $A_m\frac{dR}{da_m} + A_{m-1}\frac{dR}{da_{m-1}} + \&c. = 0$. This relation must be identical with the relation $A_m\alpha^m + A_{m-1}\alpha^{m-1} + \&c. = 0$, which we have seen is the only relation that the increments need satisfy. It follows then that the several differential coefficients are proportional to α^m, α^{m-1}, &c., and therefore that α can be found by taking the quotient of any two consecutive differential coefficients.

Cor. 1. If a_p, a_q be any two coefficients in ϕ, we must have, when $R=0$, $\frac{dR}{da_p} : \frac{dR}{da_{p-k}} :: \frac{dR}{da_q} : \frac{dR}{da_{q-k}}$; since the quotient as well of the first by the second as of the third by the fourth will $=\alpha^k$. It follows that $\frac{dR}{da_p}\frac{dR}{da_{q-k}} - \frac{dR}{da_q}\frac{dR}{da_{p-k}}$ vanishes when $R=0$, and therefore must contain R as a factor; or, in other words, $\frac{dR}{da_p}\frac{dR}{da_q} - \frac{dR}{da_r}\frac{dR}{da_s}$ contains R as a factor if we have $p+q=r+s$.

Cor. 2. It is evident, by parity of reasoning, that the differential coefficients of the eliminant, with regard to the several coefficients in ψ, are proportional to α^n, α^{n-1}, &c.; and hence, as in the last corollary, that, when $R=0$, $\frac{dR}{da_p} : \frac{dR}{da_{p-k}} :: \frac{dR}{db_q} : \frac{dR}{db_{q-k}}$; or that $\frac{dR}{da_p}\frac{dR}{db_q} - \frac{dR}{da_r}\frac{dR}{db_s}$ contains R as a factor when we have $p+q=r+s$.

Cor. 3. Or, again, if we substitute in the second equation the values of α^n, α^{n-1}, &c. given above, we have

$$b_n \frac{dR}{da_n} + b_{n-1}\frac{dR}{da_{n-1}} + \&c. = 0$$

when $R=0$. But the left-hand side of this equation cannot contain R as a factor, for it obviously contains the coefficients of ϕ in a degree less by one than that in which R contains them. It must therefore vanish identically.

97. The results of the preceding article may be confirmed by calculating the actual values of the differential coefficients of R. We know (Art. 67) that $R=\phi(\alpha)\,\phi(\beta)\,\phi(\gamma)$ &c. But since $\phi(\alpha) = a_m\alpha^m + a_{m-1}\alpha^{m-1} + \&c.$, we have $\frac{d\phi(\alpha)}{da_p} = \alpha^p$; and therefore

$$\frac{dR}{da_p} = \alpha^p\phi(\beta)\,\phi(\gamma)\,\&c. + \beta^p\phi(\alpha)\,\phi(\gamma)\,\&c. + \&c.$$

If then α satisfies ϕ, we have $\phi(\alpha)=0$, and $\frac{dR}{da_p} = \alpha^p\phi(\beta)\,\phi(\gamma)$ &c. In like manner $\frac{dR}{da_q} = \alpha^q\phi(\beta)\,\phi(\gamma)$ &c.; and therefore, as before,

$$\frac{dR}{da_p} : \frac{dR}{da_q} :: \alpha^p : \alpha^q.$$

Also, in general, if we multiply together, we find

$$\frac{dR}{da_p}\frac{dR}{da_q} = \alpha^{p+q}\{\phi(\beta)\}^2\{\phi(\gamma)\}^2\,\&c. + R(\alpha^p\beta^q + \alpha^q\beta^p)\,\phi(\gamma)\,\&c. + \&c.;$$

and it can easily be seen that the series of terms multiplying R is $\frac{d^2R}{da_p da_q}$. If now we subtract $\frac{dR}{da_r}\frac{dR}{da_s}$, the terms not multiplied by R will destroy each other if we have $p+q=r+s$, and there

will then remain

$$\frac{dR}{da_p}\frac{dR}{da_q} - \frac{dR}{da_r}\frac{dR}{da_s} = R\left(\frac{d^2R}{da_p da_q} - \frac{d^2R}{da_r da_s}\right).$$

By a similar process we can shew that $\frac{dR}{da_p}\frac{dR}{db_q} - \frac{dR}{da_q}\frac{dR}{db_p}$ is divisible by R, but the quotient is not $\frac{d^2R}{da_p db_q} - \frac{d^2R}{da_q db_p}$.

98. What has been said is applicable, as we shall presently see, to a system of equations in any number of variables. The following simpler method only applies to a system of two equations. In this case we have seen (Art. 84) that the eliminant can be expressed in the form of the determinant resulting from the elimination of x^{m-1}, x^{m-2}, &c. from a system of equations linear in these quantities. When this determinant vanishes, the equations are consistent with each other, and if we leave out any one of them, the remainder will suffice to determine x. Hence if β_{11}, β_{12}, &c. represent the minors of the determinant in question, we have x^{m-1}, x^{m-2}, &c. severally proportional to β_{11}, β_{12}, β_{13}, &c., or to β_{21}, β_{22}, β_{23}, &c., &c. These values are simpler than those found by the preceding method, since they are a degree lower than the eliminant in the coefficients of *each* equation; whereas the values found by differentiating the eliminant are a degree lower than it only in the coefficients of one of the equations. For example, the common value which satisfies the pair of equations

$$ax^2 + bx + c = 0, \quad a'x^2 + b'x + c' = 0$$

is by this method found to be $-\frac{(bc')}{(ac')} = -\frac{(ac')}{(ab')}$; whereas by the preceding method it is given in the less simple form

$$\frac{2c'(ac') - b'(bc')}{a'(bc') - c'(ab')} = \frac{a'(bc') - c'(ab')}{-2a'(ac') + b'(ab')}.$$

All these values are equal in virtue of the relation, which is supposed to be satisfied, $(ac')^2 = (ab')(bc')$.

99. If we substitute in any of the equations used in the last article the values $\frac{dR}{da_{m-1}}$ for x^{m-1}, &c., this equation must be

satisfied when $R=0$, and therefore the result of substitution must be divisible by R. In other words, if α_{r_1}, α_{r_2}, &c. be the constituents of any of the lines of the determinant of Art. 84, we must have $\alpha_{r_1}\frac{dR}{da_{m-1}}+\alpha_{r_2}\frac{dR}{da_{m-2}}+$ &c. divisible by R. But if we examine what α_{r_1}, &c. are, we see that α_{r_1} is the determinant $(a_m b_{n-r})$, &c., and thus that the function $\alpha_{r_1}\frac{dR}{da_{m-1}}+$ &c. contains the b coefficients in a degree one higher than R, while its weight exceeds that of R by $n-r+1$. Consequently the remaining factor must be b_{n-r+1} multiplied by a numerical coefficient. To determine this coefficient, we suppose all the terms of ψ to vanish except b_{n-r+1}. Now it follows at once from the method of elimination by symmetric functions, that if ψ consist of factors V, W, &c., the eliminant of ϕ and ψ is the product of the eliminants of ϕ, V; ϕ, W; &c. For if V be $(x-\alpha)(x-\beta)$, &c., and W be $(x-\alpha')(x-\beta')$, &c., the eliminant of ϕ and V is $\phi(\alpha)\,\phi(\beta)$ &c., that of ϕ and W is $\phi(\alpha')\,\phi(\beta')$ &c., and the product of all these is the eliminant of ϕ and ψ.

Again, if ψ reduce to the single term $b_\alpha x^\alpha y^\beta$, since the eliminant of ϕ and x is a_0 and of ϕ and y is a_m, the eliminant of ϕ and ψ will be $b_\alpha{}^m a_0{}^\alpha a_m{}^\beta$. The only one then of the series of terms $\frac{dR}{da_{m-1}}$, &c., which will not vanish when all the coefficients of ψ, except b_α, are made to vanish, will be $\frac{dR}{da_0}$, and this will be $\alpha b_\alpha{}^m a_0{}^{\alpha-1} a_m{}^\beta$. But in the case we are considering, it will be found that the term by which $\frac{dR}{da_0}$ is multiplied will be $b_\alpha a_0$, and hence that in general, when $\alpha=n-r+1$,

$$\alpha_{r_1}\frac{dR}{da_{m-1}}+\alpha_{r_2}\frac{dR}{da_{m-2}}+\text{&c.}=(n-r+1)\,Rb_{n-r+1}.$$

Ex. In order to make what has been said more intelligible, we repeat the proof for the particular case of the two cubics $a_3x^3+a_2x^2+a_1x+a_0$, $b_3x^3+b_2x^2+b_1x+b_0$, then we have the system of equations (Art. 84)

$$\begin{aligned}
(a_3b_2)\,x^2+\quad &(a_3b_1)\quad x+(a_2b_0)=0,\\
(a_3b_1)\,x^2+\{&(a_3b_0)+(a_2b_1)\}\,x+(a_2b_0)=0,\\
(a_3b_0)\,x^2+\quad &(a_2b_0)\quad x+(a_1b_0)=0.
\end{aligned}$$

Substituting then, suppose in the second equation, the following quantity must be divisible by R,

$$(a_3b_1)\frac{dR}{da_2}+\{(a_3b_0)+(a_2b_1)\}\frac{dR}{da_1}+(a_2b_0)\frac{dR}{da_0}.$$

But, considering the order and weight of the function in question, it is seen at once that the remaining factor must be b_2 multiplied by a numerical coefficient. To determine that coefficient, let b_0, b_1, b_3 all vanish, then the quantity we are discussing reduces to $-b_2\left(a_1\frac{dR}{da_1}+a_0\frac{dR}{da_0}\right)$. But R, on the same supposition, reduces to $b_2{}^3a_3a_0{}^2$; and therefore the function we are calculating at most differs in sign from $2b_2R$.

100. There is no difficulty in applying the method of Arts. 96, 97 to the case of any number of variables. For greater clearness we confine ourselves to three variables, but the same proof applies word for word to any number. Let there be three equations $\phi=0$, $\psi=0$, $\chi=0$, where $\phi=a_{m,0,0}x^m+\ldots+a_{\alpha,\beta,\gamma}x^\alpha y^\beta z^\gamma+$ &c., and let the values $x'y'z'$ satisfy all the equations; then they will still satisfy them if in ϕ we alter $a_{m,0,0}$, $a_{\alpha,\beta,\gamma}$ into $a_{m,0,0}+A_{m,0,0}$, $a_{\alpha,\beta,\gamma}+A_{\alpha,\beta,\gamma}$, &c, provided only that $A_{m,0,0}x'^m+$ &c. $+A_{\alpha,\beta,\gamma}x'^\alpha y'^\beta z'^\gamma+$ &c. $=0$. But, as in Art. 96, the equation must also be satisfied

$$A_{m,0,0}\frac{dR}{da_{m,0,0}}+A_{\alpha,\beta,\gamma}\frac{dR}{da_{\alpha,\beta,\gamma}}+\&c.=0;$$

and comparing these two equations, we see that the value of each term $x'^\alpha y'^\beta z'^\gamma$ must be proportional to the differential of the eliminant with respect to the coefficient which multiplies it. We obtain the values of x', y', z', by taking the ratios of the differentials of R with respect to the coefficients of any terms which are in the ratio of x, y, z. And this may be verified as in Art. 97. For let the common roots of ψ, χ, substituted in ϕ, give results ϕ', ϕ'', &c. Then $R=\phi'\phi''\phi'''$ &c. And

$$\frac{dR}{da_{\alpha,\beta,\gamma}}=x'^\alpha y'^\beta z'^\gamma\phi''\phi'''\ \&c.+x''^\alpha y''^\beta z''^\gamma\phi'\phi'''\ \&c.+\&c.;$$

and if we suppose ϕ' to vanish, the value of this differential coefficient reduces to its first term, and it is seen, as before, that the differential with regard to each coefficient is proportional to the term which that coefficient multiplies. The same corollaries may be drawn as in Art. 96.

101. And generally, in like manner, if the coefficients of ϕ be functions of any quantities a, b, c, &c., which do

not enter into ψ, χ, it is proved by the same method that $\frac{dR}{da} : \frac{dR}{db} :: \frac{d\phi}{da} : \frac{d\phi}{db}$, where in the latter differentials x, y, z are supposed to have the values x', y', z', which satisfy all the equations. For either, as in Art. 97, we have when $\phi' = 0$, $\frac{dR}{da} = \frac{d\phi}{da}\phi''\phi'''$ &c. $\frac{dR}{db} = \frac{d\phi}{db}\phi''\phi'''$ &c.; or, again, as in Art. 92, if a, b, c be varied, so that the same system of values continues to satisfy ϕ, we have

$$\frac{d\phi}{da}\delta a + \frac{d\phi}{db}\delta b + \frac{d\phi}{dc}\delta c + \&c. = 0;$$

while, because in this case the eliminant of the transformed ϕ and of the other equations continues to vanish, we have

$$\frac{dR}{da}\delta a + \frac{dR}{db}\delta b + \frac{dR}{dc}\delta c + \&c. = 0,$$

and these two equations must be identical.

102. The formulæ become more complicated if we take the differentials of the eliminant with respect to quantities a, b, &c. which enter into all the equations. As before, if we give these quantities variations, consistent with the supposition that the eliminant still vanishes, we have

$$\frac{dR}{da}\delta a + \frac{dR}{db}\delta b + \frac{dR}{dc}\delta c + \&c. = 0.$$

Now, in the former case, where a, b, c, &c. only entered into one of the equations, a change in these quantities produced no change in the value of the common roots, since the coefficients remained constant in the other equations, whose system of common roots was therefore fixed and determinate. But this will now no longer be the case, and the common roots of the transformed equations may be different from those of the original system. Let the new system of common roots be $x' + \delta x'$, $y' + \delta y'$, $z' + \delta z'$, &c., then the variations are connected by the relations

$$\frac{d\phi}{da}\delta a + \frac{d\phi}{db}\delta b + \&c. + \frac{d\phi}{dx}\delta x' + \frac{d\phi}{dy}\delta y' + \&c. = 0,$$

$$\frac{d\psi}{da}\delta a + \frac{d\psi}{db}\delta b + \&c. + \frac{d\psi}{dx}\delta x' + \frac{d\psi}{dy}\delta y' + \&c. = 0, \ \&c.$$

If there are k such equations, there will be $k-1$ independent variables;* we may, therefore, between these k relations eliminate the $k-1$ variations $\delta x'$, $\delta y'$, &c., and so arrive at a relation between the variations δa, δb, &c. only; the coefficients of which must be severally proportional to $\frac{dR}{da}$, $\frac{dR}{db}$, &c.

Ex. 1. Let there be two equations and one variable. The final relation then is

$$\left(\frac{d\phi}{da}\frac{d\psi}{dx} - \frac{d\psi}{da}\frac{d\phi}{dx}\right)\delta a + \left(\frac{d\phi}{db}\frac{d\psi}{dx} - \frac{d\psi}{db}\frac{d\phi}{dx}\right)\delta b + \&c. = 0,$$

and the several coefficients are proportional to $\frac{dR}{da}$, $\frac{dR}{db}$, &c. If the equation had been given in the homogeneous form, we might have taken x as constant, and substituted $\frac{d\phi}{dy}$, $\frac{d\psi}{dy}$ for $\frac{d\phi}{dx}$, $\frac{d\psi}{dx}$ in the preceding formula. This makes no change, because it was proved, Art. 89, that the common root satisfies the Jacobian, or makes

$$\frac{d\phi}{dx}\frac{d\psi}{dy} = \frac{d\phi}{dy}\frac{d\psi}{dx}.$$

Ex. 2. If there are three equations, the coefficient of δa is

$$\begin{vmatrix} \frac{d\phi}{da}, & \frac{d\psi}{da}, & \frac{d\chi}{da} \\ \phi_1, & \psi_1, & \chi_1 \\ \phi_2, & \psi_2, & \chi_2 \end{vmatrix},$$

where ϕ_1, ϕ_2 denote the differential coefficients of ϕ with respect to x and y, &c.

103. If a system of equations is satisfied by *two* common systems of values, not only will the eliminant R vanish, but also the differential of R with respect to every coefficient in either equation. For evidently the values of the differentials, given Art. 97, all vanish if both $\phi(\alpha)$ and $\phi(\beta) = 0$, or, in Art. 100, if ϕ', ϕ'' both $= 0$. In this case the actual values of the two common roots can be expressed by a quadratic equation in terms of the second differentials of R. The following, though for brevity, stated only for the case of two equations, applies word for word in general. We have (see Art. 97)

$$\tfrac{1}{2}\frac{d^2R}{da_p^2} = \alpha^p\beta^p\phi(\gamma)\ \phi(\delta)\ \&c. + \beta^p\gamma^p\phi(\alpha)\ \phi(\delta)\ \&c. + \&c.,$$

* If the equations had been given as homogeneous functions of k variables, still since their ratios are all we are concerned with, we may assume any one of the variables z to be the same in all the equations, and may suppose $\delta z' = 0$.

which, when $\phi(\alpha), \phi(\beta) = 0$, reduces to the single term

$$\alpha^p \beta^p \phi(\gamma)\, \phi(\delta) \text{ &c.}$$

In like manner, in the same case,

$$\frac{d^2R}{da_p da_q} = (\alpha^p\beta^q + \alpha^q\beta^p)\, \phi(\gamma)\, \phi(\delta) \text{ &c.}, \quad \tfrac{1}{2}\frac{d^2R}{da_q^2} = \alpha^q\beta^q \phi(\gamma)\, \phi(\delta) \text{ &c.}$$

If then we solve the quadratic in $\lambda : \mu$,

$$\lambda^2 \frac{d^2R}{da_q^2} - 2\lambda\mu \frac{d^2R}{da_p da_q} + \mu^2 \frac{d^2R}{da_p^2} = 0,$$

the roots will give the ratios $\alpha^p : \alpha^q$, $\beta^p : \beta^q$.

If the equations have three common systems of values, all the second differentials of R vanish, and the common roots are found by proceeding to the third differential coefficients and solving a cubic equation.

LESSON XI.

DISCRIMINANTS.

104. BEFORE entering on the subject of discriminants, we shall explain some terms and symbols which we shall frequently find it convenient to employ. In ordinary algebra we are wholly concerned with *equations*, the object usually being to find the values of x which will make a given function $= 0$. In what follows we have little to do with equations, the most frequent subject of investigation being that on which we enter in the next Lesson: namely, the discovery of those properties of a function which are unaltered by linear transformations. It is convenient, then, to have a word to denote the function itself, without being obliged to speak of the equation got by putting the function $= 0$: a word, for example, to denote $ax^2 + bxy + cy^2$ without being obliged to speak of the quadratic equation $ax^2 + bxy + cy^2 = 0$. We shall, after Prof. Cayley, use the term *quantic* to denote a homogeneous function in general; using the words quadric, cubic, quartic, quintic, n^{ic}, to denote quantics of the 2nd, 3rd, 4th, 5th, n^{th} degrees. And we distinguish quantics into binary, ternary, quaternary, n-ary, according as they contain 2, 3, 4,

n variables. Thus, by a binary cubic, we mean a function such as $ax^3+bx^2y+cxy^2+dy^3$; by a ternary quadric, such as $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, &c. Professor Cayley uses the abbreviation $(a,\ b,\ c,\ d\|x,\ y)^3$ to denote the quantic $ax^3+3bx^2y+3cxy^2+dy^3$, in which, as is usually most convenient, the terms are affected with the same numerical coefficients as in the expansion of $(x+y)^3$. So the ternary quadric written above would be expressed $(a,\ b,\ c,\ f,\ g,\ h\|x,\ y,\ z)^2$. When the terms are not thus affected with numerical coefficients, he puts an arrow-head on the parenthesis, writing, for instance $(a,\ b,\ c,\ d\|x,\ y)^3$ to denote $ax^3+bx^2y+cxy^2+dy^3$. When it is not necessary to mention the coefficients, the quantic of the n^{th} degree is written $(x,\ y)^n$, $(x,\ y,\ z)^n$, &c.

105. If a quantic in k variables be differentiated with respect to each of the variables, the eliminant of the k differentials is called the *discriminant* of the given quantic.

If n be the degree of the quantic, its discriminant is a homogeneous function of its coefficients, and is of the order $k(n-1)^{k-1}$. For the discriminant is the eliminant of k equations of the $(n-1)^{th}$ degree, and (Art. 78) must contain the coefficients of *each* of these equations in a degree equal to the product of the degrees of all the rest, that is $(n-1)^{k-1}$. And since each of these equations contains the coefficients of the original quantic in the first degree, the discriminant contains them in the $k(n-1)^{k-1}$ degree. Thus, then, the discriminant of a binary quantic is of the degree $2(n-1)$; of a ternary, is of the degree $3(n-1)^2$, &c.

106. *If in the original quantic every coefficient multiplying the first power of one of the variables x be affected with a suffix* 1, *every term multiplying the second power by a suffix* 2, *and so on; then the sum of the suffixes in each term of the discriminant is constant and $=n(n-1)^{k-1}$.* It was proved (Art. 78) that if every coefficient in a system of equations were affected with a suffix corresponding to the power of x which it multiplies, then the sum of the suffixes in every term of their eliminant would be equal to the product of the degrees of those equations, viz., $=mnp$ &c. Now suppose, that in the first of these equations

the suffix to the coefficient of x^0, instead of being 0, was l; that of x^1 was $l+1$, and so on; it is evident that the effect would be to increase the sum of the suffixes by l for every coefficient of the first equation which enters into the eliminant; and since (Art. 78) every term contains np &c. coefficients of the first equation, the total sum of suffixes is mnp &c. + lnp &c. $= (m+l)np$ &c. Now, in the present example, it is evident that every coefficient in the $k-1$ differentials U_2, U_3, &c.,* multiplies the same power of x as it did in the original quantic U. But in the remaining differential, U_1, every coefficient multiplies a power of x one less than in U, and the coefficient multiplying any term x^l in this differential will be marked with the suffix $l+1$, since it arose from differentiating a term x^{l+1} in the original quantic. It follows, then, that the sum of suffixes in the discriminant must $= (n-1)^k + (n-1)^{k-1} = n(n-1)^{k-1}$.

We shall briefly express the results of this and of the last article by saying that the *order* of the discriminant is $k(n-1)^{k-1}$; and its *weight*, $n(n-1)^{k-1}$. Thus for a binary quantic the weight of the discriminant is $n(n-1)$.

107. If a binary quantic contain a square factor, then, as is well known, the discriminant vanishes identically. For the two differentials must each contain that factor in the first degree, and therefore, since they have a common factor, their eliminant vanishes. In like manner, if a ternary quantic be of the form $X^2\phi + XY\psi + Y^2\chi$, where $X = ax + by + cz$, $Y = a'x + b'y + c'z$, then the discriminant must vanish, since every term in any of the differentials must contain either X or Y, and therefore the differentials have common the system of roots derived from the equations $X=0$, $Y=0$. In like manner, the discriminant of a quaternary quantic vanishes, if the quantic can be expressed as a function in the second degree of X, Y, Z, these being any linear functions of the variables.† We shall call those values

* We write, as before, U_1, U_2, U_3, &c. to denote the differential coefficients of U with respect to x, y, z, &c.

† In other words, the vanishing of the discriminant of an algebraical equation expresses the condition that the equation shall have equal roots; and the vanishing of the discriminant of the equation of a curve or surface expresses the condition that the curve or surface shall have a double point.

which make all the differentials vanish, the *singular roots* of the quantic.

108. We shall now discuss the properties of the discriminant of the binary quantic $U = a_0x^n + na_1x^{n-1}y + \frac{1}{2}n(n-1)a_2x^{n-2}y^2 + \&c.$

*The eliminant of U and U_1 is a_0 times the discriminant, and the eliminant of U and U_2 is a_n times the discriminant.** For since $nU = xU_1 + yU_2$, the result of substituting in nU any root of U_1 is $y'U_2'$; and when all the results of substitution are multiplied together, the product will be $y'y''y'''$ &c. (which is $= a_0$, see Art. 61), multiplied by the product of the results of substituting the same roots in U_2, which is the discriminant.

109. *To express the discriminant in terms of the values x_1y_1, x_2y_2, &c., which make the quantic vanish.*

Let $U = (xy_1 - yx_1)(xy_2 - yx_2)(xy_3 - yx_3)$ &c. (see Art. 61); then

$$U_1 = y_1(xy_2 - yx_2)(xy_3 - yx_3)\,\&c. + y_2(xy_1 - yx_1)(xy_3 - yx_3)\,\&c. + \&c.;$$

and the result of the substitution in U_1 of any root x_1y_1 of U is $y_1(x_1y_2 - y_1x_2)(x_1y_3 - y_1x_3)$ &c. Similarly, the result of substituting x_2y_2 is $y_2(x_2y_1 - x_1y_2)(x_2y_3 - y_2x_3)$ &c. If, then, all the results of substitution are multiplied together, the product is

$$\pm y_1y_2y_3\,\&c.\,(x_1y_2 - y_1x_2)^2(x_1y_3 - y_1x_3)^2(x_2y_3 - y_2x_3)^2\,\&c.$$

This, then, is the eliminant of U and U_1, and if we divide it by a_0, which is $= y_1y_2y_3$ &c., we shall have the discriminant $= (x_1y_2 - y_1x_2)^2(x_1y_3 - y_1x_3)^2$ &c. If we make in it all the y's $= 1$, we get the theorem in the well-known form that the discriminant is equal to the product of the squares of all the differences of any two roots of the equation. We shall, for simplicity, refer to the theorem in the latter form.

110. *The discriminant of the product of two quantics is equal to the product of their discriminants multiplied by the square of their eliminant.* For the product of the squares of differences of all the roots evidently consists of the product of the squares of differences of two roots both belonging to the same quantic,

* We do not take account of mere numerical factors.

multiplied by the square of the product of all differences between a root of one and a root of the other, and this latter product is the eliminant (Art. 68). As a particular case of this, the discriminant of $(x-\alpha)\,\phi(x)$ is the discriminant of $\phi(x)$ multiplied by the square of $\phi(\alpha)$. For if β, γ, &c. be the roots of $\phi(x)$, then $(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2$ &c. is equal to the square of $(\alpha-\beta)(\alpha-\gamma)$ &c. which is $\phi(\alpha)$, multiplied by the product of the squares of all differences not containing α.

111. *The discriminant of* $(a_0, a_1 \ldots a_{n-1}, a_n)(x, y)^n$ *is of the form* $a_n\phi + a_{n-1}^2\psi$, *where* ψ *is the discriminant of the equation of the* $(n-1)^{th}$ *degree* $(a_0, a_1 \ldots a_{n-2}, a_{n-1})(x, y)^{n-1}$. For we evidently get the same result, whether we put any term $a_n = 0$ in the discriminant, or first put $a_n = 0$ in the quantic, and then form the discriminant. But if we make $a_n = 0$ in the quantic, we get x multiplied by the $(n-1)^{ic}$ written above, and (Art. 110) its discriminant will then be the discriminant of that $(n-1)^{ic}$ multiplied by the square of the result of making in it $x = 0$; that is, by the square of a_{n-1}. In like manner we see that the discriminant is of the form $a_0\phi + a_1^2\psi$.*

112. The discriminant being a function of the determinants $x_1y_2 - x_2y_1$, &c must satisfy the two differential equations (Art. 62),

$$na_0\frac{d\Delta}{da_1} + (n-1)\,a_1\frac{d\Delta}{da_2} + (n-2)\,a_2\frac{d\Delta}{da_3} + \&c. = 0,$$

$$a_1\frac{d\Delta}{da_0} + 2a_2\frac{d\Delta}{da_1} + 3a_3\frac{d\Delta}{da_2} + \&c. = 0;$$

or, if the original equation had been written with binomial coefficients,

$$na_1\frac{d\Delta}{da_0} + (n-1)\,a_2\frac{d\Delta}{da_1} + \&c. = 0, \quad a_0\frac{d\Delta}{da_1} + 2a_1\frac{d\Delta}{da_2} + \&c. = 0.$$

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* This theorem was first published by Joachimsthal; I had, however, previously been led by simple geometrical considerations to the following theorem in which it is included. If a_1 contain a factor z, and if a_0 contain z^2 as a factor, the discriminant will be divisible by z^2. If a_2 contain z as a factor, if a_1 contain z^2, and a_0 contain z^3, the discriminant will in general be divisible by z^6. In like manner, if a_3 contain z; a_2, z^2; a_1, z^3; and a_0, z^4, the discriminant will be divisible by z^{12}, &c.

Ex. To form the discriminant of $(a_0, a_1, a_2, \ldots)(x, y)^n$, which we suppose arranged according to the powers of a_0. We know (Art. 111) that the absolute term is $a_1^2 D$, where D is the discriminant of $(a_1, a_2, \ldots)(x, y)^{n-1}$. The discriminant then is $a_1^2 D + a_0\phi + a_0^2\psi + \&c.$; operating on this with $a_1 \frac{d}{da_0} + 2a_2 \frac{d}{da_1} + 3a_3 \frac{d}{da_2} + \&c.$, we may equate separately to zero the coefficient of each power of a_0. Thus, then, the part independent of a_0 is

$$a_1\phi + 4a_1a_2D + a_1^2\left(2a_2\frac{d}{da_1} + 3a_3\frac{d}{da_2} + \&c.\right)D;$$

or, remembering that $\left(a_2\frac{d}{da_1} + 2a_3\frac{d}{da_2} + \&c.\right)D = 0$, we have

$$\phi = -4a_2D + a_1\left(a_2\frac{d}{da_2} + 2a_4\frac{d}{da_3} + \&c.\right)D,$$

and the discriminant is

$$(a_1^2 - 4a_0a_2)\,D + a_1a_0\left(a_3\frac{d}{da_2} + 2a_4\frac{d}{da_3} + \&c.\right)D + a_0^2\psi + \&c.$$

In like manner, from the coefficient of a_1 we can determine ψ, but the result does not seem simple enough to be worth writing down.

113. If the discriminant of a binary quantic vanishes, the quantic has equal roots, and the actual values of these roots can be found by a process similar to that employed in Lesson X. Let $U = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \&c.$ be a quantic whose discriminant vanishes, and having therefore a square factor $(x-\alpha)^2$. Then evidently V, where

$$V = A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \&c.$$

will also be divisible by $x-\alpha$, provided that A_0, A_1, &c. be any quantities satisfying the condition

$$A_0\alpha^n + A_1\alpha^{n-1} + A_2\alpha^{n-2} + \&c. = 0.$$

In this case then we shall have $U + \lambda V$ divisible by $x-\alpha$. Let it $= (x-\alpha)\{(x-\alpha)\,\phi(x) + \lambda\psi(x)\}$. It follows then, from Art. 111, that the discriminant of $U + \lambda V$ is the discriminant of the quantity within the brackets, multiplied by the square of the result of substituting α for x inside the brackets. But this result is $\lambda\psi(\alpha)$. We have proved then that in the case supposed, the discriminant of $U + \lambda V$ is divisible by λ^2.

But since $U + \lambda V$ is derived from U by altering a_0 into $a_0 + \lambda A_0$, &c., the discriminant of $U + \lambda V$ is derived from the discriminant of U by a like substitution, and is therefore

$$\Delta + \lambda\left(A_0\frac{d\Delta}{da} + A_1\frac{d\Delta}{da_1} + A_2\frac{d\Delta}{da_2} + \&c.\right) + \lambda^2(\&c.).$$

By hypothesis $\Delta = 0$. But the discriminant will not be divisible by λ^2 unless the coefficient of λ vanish. Now the relation thus obtained between A_0, A_1, &c. must be identical with the relation $A_0\alpha^n + A_1\alpha^{n-1} + \&c. = 0$, which we have already seen is the only relation that need be satisfied by A_0, A_1, &c. in order that the discriminant of $U + \lambda V$ may be divisible by λ^2. We must have therefore the quantities α^n, α^{n-1}, α^{n-2}, &c. respectively proportional to $\frac{d\Delta}{da_0}$, $\frac{d\Delta}{da_1}$, $\frac{d\Delta}{da_2}$, &c. Dividing any one of these terms by that consecutive to it, we get an expression for α. We may state this result: *When the discriminant vanishes, the several differential coefficients of the discriminant with respect to* a_0, a_1, *&c. are proportional to the differential coefficients of the quantic with respect to the same quantities.*

114. This result may be confirmed by forming the actual values of these differential coefficients in terms of the roots, which may be done by solving from the n equations

$$\frac{d\Delta}{da} = \frac{d\Delta}{da_1}\frac{da_1}{da} + \frac{d\Delta}{da_2}\frac{da_2}{da} + \&c.$$

We know the expressions for Δ, a_1, a_2, &c. in terms of the roots, and therefore from these n equations can find the n quantities sought. The result will be found to be

$$\frac{d\Delta}{da_n} = \Sigma(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2 \times \{(\alpha-\beta)(\alpha-\gamma) + (\alpha-\beta)(\alpha-\delta) + (\alpha-\gamma)(\alpha-\delta)\},$$

where the product of the squares of all the differences, not containing α, is multiplied by the sum of the products ($n-2$ taken together) of the differences which contain α,

$$\frac{d\Delta}{da_{n-1}} = \Sigma\alpha\ (\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2\{(\alpha-\beta)(\alpha-\gamma) + \&c.\},$$

$$\frac{d\Delta}{da_{n-2}} = \Sigma\alpha^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2\{(\alpha-\beta)(\alpha-\gamma) + \&c.\}, \&c.,$$

and the supposition $\alpha = \beta$ reduces these sums to quantities which are in the ratio 1, α, α^2, &c. As in Art. 96, it follows from the

theorem of the last article that $\frac{d\Delta}{da_p}\frac{d\Delta}{da_q} - \frac{d\Delta}{da_r}\frac{d\Delta}{da_s}$ is divisible by Δ when $p+q=r+s$. If more than two of the roots are equal to each other, all these differentials vanish identically, and we find the equal roots by proceeding to second differentials of the discriminant.

115. We know, from Art. 98, that instead of the functions in the last two articles, which are of an order in the coefficients only one lower than the discriminant, we may substitute functions of an order two lower, and possessing the same property, viz. that they vanish when more than two roots are equal, and that if two roots are equal ($\alpha=\beta$) they are to each other in the ratios 1, α, α^2, &c. If we proceed by Bezout's method of elimination (Art. 84) to eliminate between the two differentials U_1, U_2, the resulting equations of the $(n-2)^{th}$ degree, when expressed in terms of the roots, are $\Sigma(\alpha-\beta)^2(x-\gamma)(x-\delta)=0$, $\Sigma q_1(\alpha-\beta)^2(x-\gamma)(x-\delta)=0$, $\Sigma q_2(\alpha-\beta)^2$ &c. $=0$, where q_1, q_2, &c. are the sum, sum of products in pairs, &c. of all the roots except α and β.* The discriminant is then, by Bezout's method, expressed as a determinant, whose constituents are

$$\Sigma\ (\alpha-\beta)^2,\ \Sigma q_1\ (\alpha-\beta)^2,\ \Sigma q_2\ (\alpha-\beta)^2, \text{ &c.},$$
$$\Sigma q_1(\alpha-\beta)^2,\ \Sigma q_1^2\ (\alpha-\beta)^2,\ \Sigma q_1q_2(\alpha-\beta)^2, \text{ &c.},$$
$$\Sigma q_2(\alpha-\beta)^2,\ \Sigma q_1q_2(\alpha-\beta)^2,\ \Sigma q_2^2\ (\alpha-\beta)^2, \text{ &c., &c.}$$

And when the given equation has two roots equal, the first minors of this determinant will, by Art. 98, be in the ratio 1, α, α^2, &c. A somewhat simpler series, possessing the same property, is $\Sigma(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2$, $\Sigma\alpha(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2$, $\Sigma\alpha^2(\beta-\gamma)^2$ &c..

116. The following proof of the theorem of Art. 113 is applicable to the case of a quantic in any number of variables. For simplicity, we confine ourselves to the case of two independent variables, the method, which is that of Art. 102, being

* The first of these functions of degree $n-2$ is one of the series to which we are led by Sturm's process. With regard to the extension of Sturm's theorem, see Sylvester's memoir in the *Philosophical Transactions* for 1853.

equally applicable in general. Let the coefficients in U be functions of any quantities a, b, &c., and let variations be given to these quantities consistent with the supposition that the discriminant still vanishes, and therefore such that

$$\frac{d\Delta}{da}\delta a + \frac{d\Delta}{db}\delta b + \&c. = 0.$$

Now if the effect of this change in a, b, &c., is to alter the singular roots from x, y into $x+\delta x$, $y+\delta y$, since these new values satisfy U_1, U_2, U_3, &c., we must have

$$\frac{dU_1}{da}\delta a + \frac{dU_1}{db}\delta b + \&c. + \frac{dU_1}{dx}\delta x + \frac{dU_1}{dy}\delta y = 0,$$

$$\frac{dU_2}{da}\delta a + \frac{dU_2}{db}\delta b + \&c. + \frac{dU_2}{dx}\delta x + \frac{dU_2}{dy}\delta y = 0,$$

$$\frac{dU_3}{da}\delta a + \frac{dU_3}{db}\delta b + \&c. + \frac{dU_3}{dx}\delta x + \frac{dU_3}{dy}\delta y = 0.$$

Multiply these equations by x, y, z respectively and add; then since $nU = xU_1 + yU_2 + zU_3$, the coefficient of δa will be $n\frac{dU}{da}$; and since $\frac{dU_2}{dx} = \frac{dU_1}{dy}$, $\frac{dU_3}{dx} = \frac{dU_1}{dz}$, the coefficient of δx will be $(n-1)U_1$, which will vanish, since U_1 is satisfied for the singular roots. We get therefore

$$\frac{dU}{da}\delta a + \frac{dU}{db}\delta b + \&c. = 0,$$

and therefore the differentials of Δ with respect to a, b, &c. are proportional to the differentials of U with respect to the same quantities, it being understood that the x, y, z which occur in the latter differentials are the singular roots.

117. The theorem proved for binary quantics (Art. 111) may be extended to quantics in general. Let a be the coefficient of the highest power of any of the variables, b, c, d, &c., those of the terms involving the next highest power, then the discriminant is of the form

$$a\theta + (\phi, \chi, \psi, \&c.)(b, c, d, \&c.)^2.$$

Thus, for a ternary quantic, to which for greater simplicity we confine ourselves, if a be the coefficient of z^n; b, c those of

$z^{n-1}x$, $z^{n-1}y$; then if in the discriminant we make $a=0$, the remaining part will be of the form $b^2\phi + bc\psi + c^2\chi$. To prove this: first, let U be any quantic whose discriminant vanishes, V any other reduced to zero by the singular roots of U, then I say that the discriminant of $U+\lambda V$ will be divisible by λ^2. For, let $U=az^n+bz^{n-1}x+$&c., $V=Az^n+Bz^{n-1}x+$&c., then the coefficient of λ in the discriminant of $U+\lambda V$ will be $A\frac{d\Delta}{da}+B\frac{d\Delta}{db}+$&c., and (Art. 116) $\frac{d\Delta}{da}$, &c. are proportional to z^n, $z^{n-1}x$, &c. The coefficient of λ is therefore proportional to the result of substituting the singular roots in V, and therefore vanishes.

Now, in the case we are considering, the supposition of $a=0$, $b=0$, $c=0$ must make the discriminant vanish, since then all the differentials vanish for the singular roots $x=0$, $y=0$. Any other quantic V will vanish for the same values, provided only $A=0$. The general form of the discriminant then must be such that if we substitute for b, $b+\lambda B$; for c, $c+\lambda C$, &c., and then make a, b, $c=0$, the result must be divisible by λ^2; or, in other words, if we put for b, λB; for c, λC, &c., and then make $a=0$, the result is divisible by λ^2, which was the thing to be proved.

118. Concerning discriminants in general, it only remains to notice that the discriminant of a quadratic function in any number of variables is immediately expressed as a symmetrical determinant. And, conversely, from any symmetrical determinant, we may form a quadratic function which shall have that determinant for its discriminant. The simplest notation for the coefficients of a quadratic function is to use a double suffix, writing the coefficients of x^2, y^2, &c., a_{11}, a_{22}, a_{33}, &c., and those of xy, xz, &c., a_{12}, a_{13}; a_{12} and a_{21} being identical in this notation. The discriminant is then obviously the symmetrical determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \&c. \\ a_{21}, & a_{22}, & a_{23}, & \&c. \\ a_{31}, & a_{32}, & a_{33}, & \&c. \\ \&c. & & & \end{vmatrix}.$$

LESSON XII.

LINEAR TRANSFORMATIONS.

119. *Invariants.* The discriminant of a binary quantic being a function of the differences of the roots is evidently unaltered when all the roots are increased or diminished by the same quantity. Now the substitution of $x+\lambda$ for x is a particular case of the general *linear transformation*, where, in a homogeneous function, we substitute for each variable a linear function of the variables; as for example, in the case of a binary quantic where we substitute for x, $\lambda x+\mu y$, and for y, $\lambda' x+\mu' y$. It will illustrate the nature of the enquiries in which we shall presently engage if we examine the effect of this substitution on the discriminant of the binary quadratic, $ax^2+2bxy+cy^2$. When the variables are transformed, it becomes

$$a(\lambda x+\mu y)^2+2b(\lambda x+\mu y)(\lambda' x+\mu' y)+c(\lambda' x+\mu' y)^2;$$

and if we call the transformed equation $a'x^2+2b'xy+c'y^2$, we have

$$a'=a\lambda^2+2b\lambda\lambda'+c\lambda'^2,\quad c'=a\mu^2+2b\mu\mu'+c\mu'^2,$$
$$b'=a\lambda\mu+b(\lambda\mu'+\lambda'\mu)+c\lambda'\mu'.$$

It can now be verified without difficulty that

$$a'c'-b'^2=(ac-b^2)(\lambda\mu'-\lambda'\mu)^2;$$

that is to say, the discriminant of the transformed quadratic is equal to the discriminant of the given quadratic multiplied by the square of the determinant $\lambda\mu'-\lambda'\mu$, which is called the *modulus of transformation.*

120. Now, a corresponding theorem is true for the discriminant of any binary quantic. We can see *à priori* that this must be the case, for if a given quantic has a square factor, it will have a square factor still when it is transformed; so that whenever the discriminant of the given quantic vanishes,

that of the transformed must necessarily vanish too. The one must therefore contain the other as a factor. The theorem however can be formally proved as follows: Let the original quantic be $(xy_1 - yx_1)(xy_2 - yx_2)$ &c., then (Art. 109) the discriminant is $(x_1y_2 - y_1x_2)^2 (x_1y_3 - y_1x_3)^2$ &c.

Now the linear factor $(xy_1 - yx_1)$ of the given quantic becomes by transformation $y_1(\lambda X + \mu Y) - x_1(\lambda' X + \mu' Y)$, and if we write this in the form $Y_1X - X_1Y$, we shall have $Y_1 = \lambda y_1 - \lambda' x_1$, $X_1 = -\mu y_1 + \mu' x_1$. If then the transformed quantic be written as the product of the linear factors $(Y_1X - X_1Y)(Y_2X - X_2Y)$ &c., we have expressions, as above, for Y_1, X_1; Y_2, X_2, &c., in terms of y_1, x_1; y_2, x_2, &c. We can then, without difficulty, verify that

$$(Y_1X_2 - X_1Y_2) = (\lambda\mu' - \lambda'\mu)(y_1x_2 - x_1y_2).$$

It follows immediately that $(Y_1X_2 - Y_2X_1)^2 (Y_1X_3 - Y_3X_1)^2$ &c. is equal to $(y_1x_2 - x_1y_2)^2 (y_1x_3 - y_3x_1)^2$ &c. multiplied by a power of $\lambda\mu' - \lambda'\mu$ equal to the number of factors in the expression for the discriminant in terms of the roots. A corresponding theorem is true for the discriminant of a quantic in any number of variables.

What I have called Modern Algebra may be said to have taken its origin from a paper in the *Cambridge Mathematical Journal* for Nov. 1841, where Dr. Boole established the principles just stated and made some important applications of them. Subsequently Prof. Cayley proposed to himself the problem to determine *à priori* what functions of the coefficients of a given equation possess this property of *invariance*; that when the equation is linearly transformed, the same function of the new coefficients is equal to the given function multiplied by a quantity independent of the coefficients. The result of his investigations was to discover that this property of invariance is not peculiar to discriminants and to bring to light other important functions (some of them involving the variables as well as the coefficients) whose relations to the given equation are unaffected by linear transformation. In explaining this theory, even where, for brevity, we write only three variables, the reader is to understand that the

processes are all applicable in exactly the same way to any number of variables.

121. We suppose then that the variables in any homogeneous quantic in k variables are transformed by the substitution

$$x = \lambda_1 X + \mu_1 Y + \nu_1 Z + \&c.,$$
$$y = \lambda_2 X + \mu_2 Y + \nu_2 Z + \&c.,$$
$$z = \lambda_3 X + \mu_3 Y + \nu_3 Z + \&c., \&c.,$$

and we denote by Δ the *modulus of transformation;* namely, the determinant, whose constituents are the coefficients of transformation, λ_1, μ_1, ν_1, &c., λ_2, μ_2, ν_2, &c., &c.

Now it is evidently not possible in general so to choose the coefficients λ_1, μ_1, &c., that a certain given function $ax^n + \&c.$ shall assume, by transformation, another given form $a'X^n + \&c.$ In fact, if we make the substitution in $ax^n + \&c.$, and then equate coefficients, we obtain, as in Art. 119, a series of equations $a' = a\lambda_1^n + \&c.$, the number of which will be equal to the number of terms in the general function of the n^{th} degree in k variables. And to satisfy these equations we have only at our disposal the k^2 constants λ_1, λ_2, &c., a number which will in general be less than the number of equations to be satisfied.* It follows then that when a function $ax^n + \&c.$ is capable of being transformed into $a'X^n + \&c.$, there will be relations connecting the coefficients a, b, &c., a', b', &c. In fact, we have only to eliminate the k^2 constants from any $k^2 + 1$ of the equations $a' = a\lambda_1^n + \&c.$, and we obtain a series of relations connecting a, a', &c., which will be equivalent to as many independent relations as the excess over k^2 of the number of equations. Thus, in the case of a binary quantic, the number

* The number of terms in the general equation of the n^{th} degree homogeneous in k variables is $\frac{(n+1)(n+2)\ldots(n+k-1)}{1.2\ldots(k-1)}$; and it is easy to see that the only cases where this number is not greater than k^2 are, first, when $n = 2$, when it becomes $\frac{1}{2}k(k+1)$, a number necessarily less than k^2, k being an integer; and secondly, the case $k = 2$, $n = 3$, when both numbers have the same value 4. That is to say, the only cases where a given function can be made by transformation to assume any assigned form are, first, the case of a quadratic function in any number of variables; and secondly, the case of a cubic function homogeneous in two variables.

of terms in a homogeneous function of the n^{th} degree is $n+1$. If then, in any quantic ax^n + &c., we substitute for x, $\lambda_1 X + \mu_1 Y$, and for y, $\lambda_2 X + \mu_2 Y$, and if we then equate coefficients with $a'X^n$ + &c., we have $n+1$ equations connecting a, a', λ_1, &c., from which, if we eliminate the four quantities λ_1, λ_2, μ_1, μ_2, we get a system equivalent to $n-3$ independent relations between a, b, a', b', &c. It will appear in the sequel that these relations can be thrown into the form $\phi(a, b, \&c.) = \phi(a', b', \&c.)$; or, in other words, that there are functions of the coefficients a, b, &c. which are equal to the same functions of the transformed coefficients. The process indicated in this article is not that which we shall actually employ in order to find such functions, but it gives an *à priori* explanation of the existence of such functions, and it shows what number of such functions, independent of each other, we may expect to find.

122. Any function of the coefficients of a quantic is called an *invariant*, if, when the quantic is linearly transformed, the same function of the new coefficients is equal to the old function multiplied by some power of the modulus of transformation; that is to say, when we have

$$\phi(a', b', c', \&c.) = \Delta^p \phi(a, b, c, \&c.).$$

Such a function is said to be an *absolute invariant* when $p=0$; that is to say, when the function is absolutely unaltered by transformation even though Δ be not $=1$. If a quantic have two ordinary invariants, it is easy to deduce from them an absolute invariant. For if it have an invariant ϕ, which when transformed becomes multiplied by Δ^p, and another ψ, which when transformed becomes multiplied by Δ^q, then evidently the q^{th} power of ϕ divided by the p^{th} power of ψ will be a function which will be absolutely unchanged by transformation.

It follows, from what has been just said, that a binary quadratic or cubic can have no invariant but the discriminant, which we saw (Art. 120) is an invariant. For if there were a second, we could from the two deduce a relation $\phi(a, b, \&c.) = \phi(a', b', \&c.)$. But we see from Art. 121 that there can be no relation connecting a, b, &c. with a', b', &c.,

since, with the help of the four constants λ_1, &c. at our disposal, we can transform a given quadratic or cubic, so that the coefficients of the transformed equation may have any values we please. In the same manner we see that a quantic of the second order in any number of variables can have no invariant but the discriminant. On the other hand, suppose we take the binary quartic $ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$, and that the coefficients become by linear transformation a', b', &c., it will be found that we have two invariant functions both distinct from the discriminant; in fact, we have the two equations

$$a'e' - 4b'd' + 3c'^2 = \Delta^4 (ae - 4bd + 3c^2),$$

$$a'c'e' + 2b'c'd' - a'd'^2 - e'b'^2 - c'^3 = \Delta^6 (ace + 2bcd - ad^2 - eb^2 - c^3),$$

and from these two we deduce the absolute invariant

$$\frac{(a'c'e' + 2b'c'd' - a'd'^2 - e'b'^2 - c'^3)^2}{(a'e' - 4b'd' + 3c'^2)^3} = \frac{(ace + 2bcd - ad^2 - eb^2 - c^3)^2}{(ae - 4bd + 3c^2)^3}.$$

In this case the invariance of the discriminant may be deduced as a consequence of the preceding equations, for the discriminant is

$$(ae - 4bd + 3c^2)^3 - 27 (ace + 2bcd - ad^2 - eb^2 - c^3)^2,$$

and consequently the discriminant of the transformed equation is equal to that of the original multiplied by Δ^{12}.

123. In the same manner as we have invariants of a single quantic we may have invariants of a system of quantics. Let there be any number of simultaneous equations $ax^n + \&c. = 0$, $a'x^n + \&c. = 0$, &c., and if when the variables in all are transformed by the same substitution, these become $AX^n + \&c. = 0$, $A'X^n + \&c. = 0$, &c., then any function of the coefficients is an invariant if the same function of the new coefficients is equal to the old function multiplied by a power of the modulus of transformation; that is to say, if

$$\phi (A, B, \&c., A', B', \&c., A'', \&c.) = \Delta^r \phi (a, b, \&c., a', b', \&c., a'', \&c.).$$

The simplest example of such invariants is the case of a system of linear equations. The determinant of such a system

is an invariant of the system. This is evident at once from the definition of an invariant and from the form in which the fundamental theorem for the multiplication of determinants has been stated in Art. 23.

If we are given an invariant of a single quantic, we can derive from it a series of invariants of systems of quantics of the same degree. In order to make the spirit of the method more clear, we illustrate it in the first instance by a simple example. We have seen (Art. 119) that $ac-b^2$ is an invariant of the quadratic $ax^2+2bxy+cy^2$, and we shall now thence derive an invariant of a system of two quadratics. Suppose that by a linear transformation $ax^2+2bxy+cy^2$ becomes $AX^2+2BXY+CY^2$, and $a'x^2+2b'xy+c'y^2$ becomes $A'X^2+2B'XY+C'Y^2$; then evidently, by the same transformation (k being any constant),

$$(a+ka')\,x^2+2\,(b+kb')\,xy+(c+kc')\,y^2$$

will become

$$(A+kA')\,X^2+2\,(B+kB')\,XY+(C+kC')\,Y^2.$$

Forming then the invariant of the last quadratic, we have (Art. 119)

$$(A+kA')\,(C+kC')-(B+kB')^2=\Delta^2\{(a+ka')(c+kc')-(b+kb')^2\}.$$

But since k is arbitrary, the coefficients of the respective powers of k must be equal on both sides of the equation; and therefore we have not only, as we knew before,

$$(AC-B^2)=\Delta^2\,(ac-b^2),\quad (A'C'-B'^2)=\Delta^2\,(a'c'-b'^2),$$

but also $$AC'+CA'-2BB'=\Delta^2\,(ac'+ca'-2bb'),$$

an equation which may also be directly verified by the values of A, B, &c. given Art. 119. We see then that $ac'+ca'-2bb'$ is an invariant.

By exactly the same method, if we have any invariant of a quantic ax^n+&c., and if we want to form invariants of the system ax^n+&c., $a'x^n+$&c., we have only to substitute in the given invariant for each coefficient a, $a+ka'$, for b, $b+kb'$, &c., and the coefficient of each power of k in the result will be an invariant. Writing down, by Taylor's theorem, the result of

substituting $a + ka'$ for a, &c., the theorem to which we have been led may be stated thus: If we have any invariant of a quantic $ax^n + \&c$, and if we perform on it the operation $a' \frac{d}{da} + b' \frac{d}{db} + \&c.$, we get an invariant of the system of two quantics $ax^n + \&c.$, $a'x^n + \&c.$ We may repeat the same operation and thus get another invariant of the system, or we may operate with $a'' \frac{d}{da} + b'' \frac{d}{db} + \&c.$, and thus get an invariant of a system of three quantics, and so on. This latter process gives us the invariants which we should find by substituting for a, $a + ka' + la''$, &c., and taking the coefficients of the products of every power of k and l. In the same manner we get invariants of a system of any number of quantics.

124. *Covariants.* A *covariant* is a function involving not only the coefficients of a quantic, but also the variables, and such that when the quantic is linearly transformed, the same function of the new variables and coefficients shall be equal to the old function multiplied by some power of the modulus of transformation; that is to say, if $ax^n + \&c.$ when transformed becomes $AX^n + \&c.$, a function ϕ will be a covariant* if it is such that

$$\phi(A, B, \&c., X, Y, \&c.) = \Delta^r \phi(a, b, \&c., x, y, \&c.).$$

Every invariant of a covariant is an invariant of the original quantic. This follows at once from the definitions. Let the quantic be $ax^n + \&c.$, and the covariant $a'x^m + \&c.$ which are supposed to become by transformation $AX^n + \&c.$, $A'X^m + \&c.$ Now an invariant of the covariant is a function of its coefficients such that

$$\phi(A', B', \&c.) = \Delta^r \phi(a', b', \&c.).$$

* In the geometry of curves and surfaces, all transformations of coordinates are effected by linear substitution. An invariant of a ternary or quaternary quantic is a function of the coefficients, whose vanishing expresses some property of the curve or surface independent of the axes to which it is referred, as, for instance, that the curve or surface should have a double point. A covariant will denote another curve or surface, the locus of a point whose relation to the given curve is independent of the choice of axes. Hence the geometrical importance of the theory of invariants and covariants.

But A', B', &c. by definition can only differ by a power of the modulus from being the same functions of A, B, &c. that a', b', &c. are of a, b, &c. Hence when the functions are both expressed in terms of the coefficients of the original quantic and its transformed, we have

$$\psi(A, B, \&c.) = \Delta^t \psi(a, b, \&c.),$$

or the function is an invariant. Similarly, a covariant of a covariant is a covariant of the original quantic.

125. We shall in this and the next article establish principles which lead to an important series of covariants.

If in any quantic u we substitute $x + kx'$ for x, $y + ky'$ for y, &c., where $x'y'z'$ are cogredient to xyz, then the coefficients of the several powers of k, which are all of the form $\left(x' \frac{d}{dx} + y' \frac{d}{dy} + \&c.\right)^p u$, have been called the first, second, third, &c. *emanants** of the quantic. Now *each of these emanants is a covariant of the quantic.* We evidently get the same result whether in any quantic we write $x + kx'$ for x, &c., and then transform x, x', &c. by linear substitutions, or whether we make the substitutions first and then write $X + kX'$ for X, &c. For plainly

$$\lambda_1 X + \mu_1 Y + \nu_1 Z + k(\lambda_1 X' + \mu_1 Y' + \nu_1 Z')$$
$$= \lambda_1 (X + kX') + \mu_1 (Y + kY') + \nu_1 (Z + kZ').$$

If then u becomes by transformation U, we have proved that the result of writing $x + kx'$ for x, &c. in u, must be the same as the result of writing $X + kX'$ for X, &c. in U, and since k is indeterminate, the coefficients of k must be equal on both sides of the equation; or

$$x' \frac{du}{dx} + y' \frac{du}{dy} + \&c. = X' \frac{dU}{dX} + Y' \frac{dU}{dY} + \&c., \&c. \quad \text{Q.E.D.}$$

126. *If we regard any emanant as a function of x', y', &c., treating x, y, &c. as constants, then any of its invariants will be*

* In geometry emanants denote the polar curves or surfaces of a point with regard to a curve or surface.

a covariant of the original quantic when x, y, &c. are considered as variables.

We have just seen that $x'^p \frac{d^p u}{dx^p} + \&c.$ becomes $X'^p \frac{d^p U}{dX^p} + \&c.$ when we substitute for x', $\lambda_1 X' + \mu_1 Y' + \&c.$, and for x, $\lambda_1 X + \mu_1 Y + \&c.$ It is evidently a matter of indifference whether the substitutions for x', &c., and for x, &c., are simultaneous or successive. If then on transforming x', &c. alone, $x'^p \frac{d^p u}{dx^p} + \&c.$ becomes $aX'^p + \&c.$, then a, &c. will be such functions of x, &c. as when x, y, &c. are transformed will become $\frac{d^p U}{dX^p}$, &c. Now an invariant of the given emanant considered as a function of x', y', &c. only, is by definition such a function of its coefficients as differs only by a power of the modulus from the corresponding function of the transformed coefficients a, b, &c. But since, as we have seen, a, &c. become $\frac{d^p U}{dX^p}$, &c. when x, &c. are transformed, it follows that the given invariant will be a function of $\frac{d^p u}{dx^p}$, &c., which when x, &c. are transformed will differ only by a power of the modulus from the corresponding function of $\frac{d^p U}{dX^p}$, &c. It is therefore by definition a covariant of the quantic.

Thus then, for example, since we have proved (Art. 119) that if the binary quantic $ax^2 + 2bxy + cy^2$ becomes by transformation $AX^2 + 2BXY + CY^2$, then

$$AC - B^2 = \Delta^2 (ac - b^2);$$

it follows now, by considering the second emanant $\left(x' \frac{d}{dx} + y' \frac{d}{dy}\right)^2 u$ of a quantic of any degree, that

$$\frac{d^2 U}{dX^2} \cdot \frac{d^2 U}{dY^2} - \left(\frac{d^2 U}{dX dY}\right)^2 = \Delta^2 \left\{\frac{d^2 u}{dx^2} \cdot \frac{d^2 u}{dy^2} - \left(\frac{d^2 u}{dx\, dy}\right)^2\right\},$$

a theorem of which other demonstrations will be given.

127. In general, if we take the second emanant of a quantic in any number of variables, and form its discriminant, this will

be a covariant which is called the *Hessian* of the quantic. It was noticed (Art. 118) that the discriminant of every quadratic function may be written as a determinant. Thus then if, as we have done elsewhere, we use the suffixes 1, 2, &c. to denote differentiation with respect to x, y, &c., so that, for example, u_{11} shall denote $\frac{d^2u}{dx^2}$, then the quadratic emanant is $u_{11}x'^2 + 2u_{12}x'y' + \&c.$, and its discriminant, which is the Hessian, is the determinant

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & \&c. \\ u_{21}, & u_{22}, & u_{23}, & \&c. \\ u_{31}, & u_{32}, & u_{33}, & \&c. \\ \&c. & & & \end{vmatrix}$$

128. We have seen (Art. 123) that the determinant of a system of linear equations is an invariant of the system. If then, given a system u, v, w, &c. of as many functions as variables, we take the first emanants

$$x'u_1 + y'u_2 + z'u_3 + \&c., \&c.,$$

their determinant

$$\begin{vmatrix} u_1, & u_2, & u_3, & \&c. \\ v_1, & v_2, & v_3, & \&c. \\ w_1, & w_2, & w_3, & \&c. \\ \&c. & & & \end{vmatrix}$$

is a covariant of the system. This is the determinant already called the Jacobian (Art. 88). The Hessian is the Jacobian of the system of differentials of a single quantic u_1, u_2, u_3, &c.

129. *Contravariants.* When a set of variables x, y, &c. are linearly transformed, it constantly happens that other variables connected with them are also linearly transformed, but by a substitution different from that which is applied to x, y, &c. If the equations connecting x, y, z with the new variables be written as before

$$x = \lambda_1 X + \mu_1 Y + \nu_1 Z, \quad y = \lambda_2 X + \mu_2 Y + \nu_2 Z, \quad z = \lambda_3 X + \mu_3 Y + \nu_3 Z,$$

then variables ξ, η, ζ are said to be transformed by the *inverse* substitution, if the new variables, expressed in terms of the

old, are

$$\Xi = \lambda_1\xi + \lambda_2\eta + \lambda_3\zeta,\quad \mathrm{H} = \mu_1\xi + \mu_2\eta + \mu_3\zeta,\quad \mathrm{Z} = \nu_1\xi + \nu_2\eta + \nu_3\zeta,$$

where if in the first substitution the coefficients are the constituents of the determinant $(\lambda_1\mu_2\nu_3)$ read horizontally, in the second they are the same constituents read vertically; and where if in the first substitution the old variables are expressed in terms of the new, in the second the new are expressed in terms of the old. Stated thus, it is evident that the relation between the two substitutions is reciprocal. Solving for ξ, η, ζ in terms of Ξ, H, Z, we get (Art. 29)

$$\Delta\xi = L_1\Xi + M_1\mathrm{H} + N_1\mathrm{Z},\quad \Delta\eta = L_2\Xi + M_2\mathrm{H} + N_2\mathrm{Z},$$
$$\Delta\zeta = L_3\Xi + M_3\mathrm{H} + N_3\mathrm{Z},$$

where L_1, M_1, &c. are the minors obtained by striking out from the matrix of the determinant $(\lambda_1\mu_2\nu_3)$ (the modulus of transformation) the line and column containing λ_1, μ_1, &c.

Sets of variables x, y, z; ξ, η, ζ, supposed to be transformed according to the different rules here explained, are said to be *contragredient* to each other. In what follows, variables supposed to be contragredient to x, y, z are denoted by Greek letters, the letters α, β, γ being usually employed in subsequent lessons. We proceed to explain two of the most important cases in which the inverse substitution is employed.

130. When a function of x, y, z, &c., is transformed by linear substitutions to a function of X, Y, Z, &c., then the differential coefficients, with respect to the new variables, are linear functions of those with respect to the old, but are expressed in terms of them by the *inverse* substitution. We have

$$\frac{d}{dX} = \frac{d}{dx}\frac{dx}{dX} + \frac{d}{dy}\frac{dy}{dX} + \frac{d}{dz}\frac{dz}{dX} + \&c.$$

But from the expressions for x, y, &c. in terms of X, Y, &c., we have

$$\frac{dx}{dX} = \lambda_1,\quad \frac{dy}{dX} = \lambda_2,\quad \frac{dz}{dX} = \lambda_3.$$

Hence then

$$\frac{d}{dX} = \lambda_1\frac{d}{dx} + \lambda_2\frac{d}{dy} + \lambda_3\frac{d}{dz} + \&c.$$

Similarly $\frac{d}{dY} = \mu_1 \frac{d}{dx} + \mu_2 \frac{d}{dy} + \mu_3 \frac{d}{dz} + \text{&c., &c.}$

Thus then, according to the definition given in the last article, the operating symbols $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$, &c. are *contragredient* to x, y, z, &c., that is to say, when the latter are linearly transformed, the former will be linearly transformed also, but according to the different rule explained in the last article.

If, as before, u_1, u_2, &c. denote the differential coefficients of u, and U_1, U_2, &c. those of the transformed function U, we have just proved that

$$U_1 = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3, \quad U_2 = \mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3, \text{ &c.}$$

Consequently, if u_1, u_2, u_3 all vanish, U_1, U_2, U_3 must all vanish likewise. Now we know that u_1, u_2, u_3 all vanish together only when the discriminant of the system vanishes; if then the discriminant of the original system vanishes, we see now that the discriminant of the transformed system must vanish likewise, and therefore that the latter contains the former as a factor, as has been already stated (Art. 120).

131. In plain geometry, if x, y, z be the trilinear coordinates of any point, and $x\xi + y\eta + z\zeta = 0$ be the equation of any line, ξ, η, ζ may be called the tangential coordinates of that line (see *Conics*, Art. 70). Now, if the equation be transformed to any new system of axes by the substitution $x = \lambda_1 X + \text{&c.}$, the new equation of the line becomes

$$\xi(\lambda_1 X + \mu_1 Y + \nu_1 Z) + \eta(\lambda_2 X + \mu_2 Y + \nu_2 Z) + \zeta(\lambda_3 X + \mu_3 Y + \nu_3 Z) = 0,$$

so that if the new equation of the right line be written $\Xi X + \mathrm{H} Y + \mathrm{Z} Z = 0$, we have

$$\Xi = \lambda_1 \xi + \lambda_2 \eta + \lambda_3 \zeta, \quad \mathrm{H} = \mu_1 \xi + \mu_2 \eta + \mu_3 \zeta, \quad \mathrm{Z} = \nu_1 \xi + \nu_2 \eta + \nu_3 \zeta.$$

In other words, *when the coordinates of a point are transformed by a linear substitution, the tangential coordinates of a line are transformed by the inverse substitution;* that is, they are contragredient to the coordinates of the point. In like manner, in the geometry of three dimensions, the tangential coordinates of any plane are contragredient to the coordinates of any

point. When we transform to new axes, all coordinates $xyzw$, $x'y'z'w'$, &c. expressing different points, are cogredient: that is to say, all must be transformed by the same substitution $x = \lambda_1 X + \&c.$, $x' = \lambda_1 X' + \&c.$, &c. But the tangential coordinates of every plane will be transformed by the inverse substitution, as we have just explained. Similarly the ray coordinates of different lines for the same system of reference are cogredient, but the axial coordinates are transformed by the inverse substitution, that is, are contragredient to the former. See *Surfaces*, Art. 57*e*.

The principle just stated will be frequently made use of in the form

$$x\xi + y\eta + z\zeta = X\Xi + Y\mathrm{H} + Z\mathrm{Z},$$

where x, y, z being supposed to be changed by the substitution, $x = \lambda_1 X + \mu_1 Y + \&c.$, ξ, η, ζ are supposed to be changed by the inverse substitution $\Xi = \lambda_1 \xi + \lambda_2 \eta + \lambda_3 \zeta$, &c. In other words, in the case supposed, $x\xi + y\eta + z\zeta$ is a function absolutely unaltered by transformation, and analogous statements easily follow in the other cases mentioned.

132. If a function $ax^n + \&c.$ becomes by transformation $AX^n + \&c.$, then any function involving the coefficients and those variables which are supposed to be transformed by the inverse substitution is said to be a *contravariant* if it is such that it differs only by a power of the modulus from the corresponding function of the transformed coefficients and variables: that is to say, if

$$\phi(A, B, \&c., \Xi, \mathrm{H}, \&c.) = \Delta^r \phi(a, b, \&c., \xi, \eta, \&c.).$$

Such functions constantly present themselves in geometry. If we have an equation expressing the condition that a line or plane should have to a given curve or surface a relation independent of the axes to which it is referred (as, for example, the condition that the line or plane should touch the curve or surface), then, when we transform to new axes, it is obviously indifferent whether we transform the given relation by substituting for the old coefficients their values in terms of the new, or whether we derive the condition by the original rule from the transformed equation. In this way it is seen

that the conditions in question are of such a kind that $\phi(a, b, \xi, \&c.)$ differs only by a factor from $\phi(A, B, \Xi, \&c.)$.

133. Besides covariants and contravariants there are also functions involving both sets of variables, which differ only by a power of the modulus from the corresponding trans formed functions: *i.e.* such that

$$\phi(A, B, \&c., X, Y, \&c., \Xi, H, \&c.) = \Delta^r \phi(a, b, \&c., x, y, \&c., \xi, \eta, \&c.).$$

Dr. Sylvester uses the name concomitant as a general word to include all functions whose relations to the quantic are unaltered by linear transformation, and he calls the functions now under consideration *mixed concomitants*. I do not choose to introduce a name on my own responsibility; otherwise I should be inclined to call them divariants. The simplest function of the kind is $x\xi + y\eta + z\zeta$, which we have seen (Art. 131) is transformed to a similar function, and is therefore a concomitant of every quantic whatever.

134. If we are given any invariant I of the quantic

$$a_0x^n + na_1x^{n-1}y + nb_1x^{n-1}z + \tfrac{1}{2}n(n-1)\,a_2x^{n-2}y^2 + \&c.,$$

we can deduce from it a contravariant by the method used in Art. 123. If $a_0x^n + \&c.$ becomes by transformation $A_0X^n + \&c.$, then, since $x\xi + \&c.$ becomes $X\Xi + \&c.$, it follows that

$$a_0x^n + \&c. + k(x\xi + y\eta + z\zeta)^n = A_0X^n + \&c. + k(X\Xi + Y\mathrm{H} + Z\mathrm{Z})^n.$$

Now an invariant of the original quantic fulfils the condition

$$\phi(A_0, A_1, B_1, \&c.) = \Delta^r \phi(a_0, a_1, b_1, \&c.).$$

Forming then the same invariant of the new quantic, it will be seen that

$$\phi(A_0 + k\Xi^n,\ A_1 + k\Xi^{n-1}\mathrm{H}, \&c.) = \Delta^r \phi(a_0 + k\xi^n,\ a_1 + k\xi^{n-1}\eta, \&c.).$$

Since k is arbitrary we may equate the coefficients of like powers of k on both sides of this equation.

But, by Taylor's theorem, these coefficients are all of the form

$$\left(\xi^n \frac{d}{da_0} + \xi^{n-1}\eta \frac{d}{da_1} + \xi^{n-1}\zeta \frac{d}{db_1} + \xi^{n-2}\eta^2 \frac{d}{da_2} + \&c.\right)^r I.$$

We have proved then that they differ only by a power of the modulus from the corresponding function of the transformed equation. They are, therefore, *contravariants*, since it is assumed all along that ξ, η, ζ are to be transformed by the inverse substitution. Dr. Sylvester has called contravariants formed by this rule, first, second, &c. *evectants* of the given invariant. Thus $\xi^n \frac{dI}{da_0} + \xi^{n-1}\eta \frac{dI}{da_1} +$ &c. is the first evectant. It is to be observed that in the original quantic the coefficients are supposed to be written *with*, and in the evectant *without*, binomial coefficients. Comparing this article with Art. 123 we see that the function $\xi^n \frac{dI}{da_0} +$ &c. may be considered either as a *contravariant* of the single given quantic, or as an *invariant* of the system obtained by combining with the given quantic the linear function $x\xi + y\eta + z\zeta$. The theory of contravariants, therefore, may be included under that of invariants.

If we perform the operation $\xi^n \frac{d}{da_0} +$ &c. upon any covariant we obtain a mixed concomitant, for it is proved in the same way that the result, which will evidently be a function involving variables of both kinds, will be transformed into a function of similar form.

Ex. 1. We know that $ac - b^2$ is an invariant of $ax^2 + 2bxy + cy^2$; hence $c\xi^2 - 2b\xi\eta + a\eta^2$ is a contravariant of the same system.

Ex. 2. Similarly, $abc + 2fgh - af^2 - bg^2 - ch^2$, being the discriminant, and therefore an invariant of $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$,
$(bc - f^2)\xi^2 + (ca - g^2)\eta^2 + (ab - h^2)\zeta^2 + 2(gh - af)\eta\zeta + 2(hf - bg)\zeta\xi + 2(fg - ch)\xi\eta$
is a contravariant of the same quantic. Geometrically, as is well known, the function equated to zero expresses the tangential equation of the conic represented by the given quantic.

Ex. 3. Given a system of two ternary quadrics $ax^2 +$ &c., $a'x^2 +$ &c., then since $a'(bc - f^2) +$ &c. is an invariant of the system (Art. 123), we find on operating with $\xi^2 \frac{d}{da} +$ &c., that
$(bc' + b'c - 2ff')\xi^2 + (ca' + c'a - 2gg')\eta^2 + (ab' + a'b - 2hh')\zeta^2$
$+ 2(gh' + g'h - af' - a'f)\eta\zeta + 2(hf' + h'f - bg' - b'g)\zeta\xi + 2(fg' + f'g - ch' - c'h)\xi\eta$
is a contravariant of the system. We might have equally found this contravariant by operating with $a' \frac{d}{da} +$ &c. on the contravariant of the last example. Geometrically, the function equated to zero expresses the condition that a line should be cut harmonically by two conics.

135. When the discriminant of a quantic vanishes, it has a set of singular roots $x'y'z'$ [geometrically the coordinates of the double point on the curve or surface represented by the quantic]; and in this case the first evectant will be the perfect n^{th} power of $(x'\xi + y'\eta + z'\zeta)$. Since we have seen that this evectant is a function unaltered by transformation, it is sufficient to see what it becomes in any particular case. Now if the discriminant vanishes, the quantic can be so transformed that the new coefficients of x^n, $x^{n-1}y$, $x^{n-1}z$ shall vanish; that is to say, so that the singular root shall be $y=0$, $z=0$ [geometrically, so that the point yz shall be the double point]. But it was proved (Art. 117) that the form of the discriminant is

$$a_0\rho + a_1^2\phi + a_1b_1\psi + b_1^2\chi.$$

Evidently then, not only will this vanish when a_0, a_1, b_1 vanish, but also its differentials with respect to every coefficient except a_0 will vanish. This evectant then reduces itself to $\frac{dI}{da_0}$ multiplied by the perfect n^{th} power ξ^n, which is what $(x'\xi + y'\eta + z'\zeta)^n$ becomes when y' and $z'=0$, and $x'=1$. Thus then, if the discriminant of a ternary quadric vanish, the quadric represents two lines: the contravariant

$$(bc-f^2)\,\xi^2 + (ca-g^2)\,\eta^2 + \&c.$$

becomes a perfect square; and if we identify it with $(x'\xi+y'\eta+z'\zeta)^2$, we get $x'y'z'$ the coordinates of the intersection of the pair of lines. If a quantic have two sets of singular roots, all the first differentials of the discriminant vanish, and its second evectant becomes the perfect n^{th} power of

$$(x'\xi + y'\eta + z'\zeta)\,(x''\xi + y''\eta + z''\zeta),$$

where $x'y'z'$, $x''y''z''$ are the two sets of singular roots; and so on.

LESSON XIII.

FORMATION OF INVARIANTS AND COVARIANTS.

136. HAVING now shewn what is meant by invariants, &c., we go on to explain the methods by which such functions can be formed. Three of these methods will be explained in this Lesson, and a fourth in the next Lesson.

Symmetric functions. The following method is only applicable to binary quantics. *Any symmetric function of the differences of the roots is an invariant, provided that each root enters into the expression the same number of times.** It is evident that an invariant must be a function of the differences of the roots, since it is to be unaltered when for x we substitute $x+\lambda$. Now the most general linear transformation is evidently equivalent to an alteration of each root α into $\frac{\lambda\alpha+\mu}{\lambda'\alpha+\mu'}$. By this change the difference between any two roots $\alpha-\beta$ becomes $\frac{(\lambda\mu'-\lambda'\mu)(\alpha-\beta)}{(\lambda'\alpha+\mu')(\lambda'\beta+\mu')}$. In order then, that any function of the differences may, when transformed, differ only by a factor from its former value, it is necessary that the denominator should be the same for every term; and therefore the function must be a product of differences, in which each root occurs the same number of times. Thus for a biquadratic, $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$ is an invariant, because, when we transform, all the terms of which the sum is made up have the same denominator. But

* If in the equation the highest power of x is written with a coefficient a_0, we have to divide by that coefficient in order to obtain the expression for the sum, &c. of the roots; and all symmetric functions of the roots are fractions containing powers of a_0 in the denominator. When we say that a symmetric function of the roots is an invariant, we understand that it has been made integral by multiplying it by such a power of a_0 as will clear it of fractions; or, what comes to the same thing, if we form the symmetric function on the supposition that the coefficient of x^n is 1, that we make it homogeneous by multiplying each term by whatever power of a_0 may be necessary.

$\Sigma(\alpha-\beta)^2$ is not an invariant, the denominator for the term $(\alpha-\beta)^2$ being $(\lambda'\alpha+\mu')^2(\lambda'\beta+\mu')^2$, and for the term $(\gamma-\delta)^2$ being $(\lambda'\gamma+\mu')^2(\lambda'\delta+\mu')^2$.

137. Or perhaps the same thing may be more simply stated by writing the equation in the homogeneous form. We saw (Art. 120) that if we change x into $\lambda x+\mu y$, y into $\lambda' x+\mu' y$, the quantity $x_1y_2-x_2y_1$ becomes $(\lambda\mu'-\lambda'\mu)(x_1y_2-x_2y_1)$, and consequently any function of the determinants $x_1y_2-x_2y_1$ &c. is an invariant. Now (Art. 61) any function of the roots expressed in the ordinary way is changed to the homogeneous form by writing for α, β, &c. $\frac{x_1}{y_1}$, $\frac{x_2}{y_2}$, &c., and then multiplying by such a power of the product of all the y's as will clear it of fractions. If any function of the differences in which all the roots do not equally occur be treated in this way, powers of the y's will remain after multiplication, and the function will not be an invariant. Thus, for a biquadratic, $\Sigma(\alpha-\beta)^2$ becomes $\Sigma y_3^2y_4^2(x_1y_2-x_2y_1)^2$; but the function $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$, in which all the roots occur, becomes $\Sigma(x_1y_2-x_2y_1)^2(x_3y_4-x_4y_3)^2$, and this being a function of the determinants only, *is* an invariant.

It is proved in like manner, that any symmetric function formed of differences of roots and differences between x and one or more of the roots is a covariant, provided that each root enters the same number of times into the expression. Thus for a cubic $\Sigma(\alpha-\beta)^2(x-\gamma)^2$ is a covariant.

138. We can, by the method just explained, form invariants or covariants which shall vanish on the hypothesis of any system of equalities between the roots. Thus, let it be required to form an invariant which shall vanish when any three roots are all equal, it is evident that every term must contain some one of the three differences $\alpha-\beta$, $\beta-\gamma$, $\gamma-\alpha$; and in like manner for every other set of three that can be formed out of the roots. Thus, in a biquadratic, there are four sets of three roots: the difference $\alpha-\beta$ belongs to two of these sets, and $\gamma-\delta$ to the other two; therefore $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$* is an invariant which

* $\Sigma(\alpha-\beta)(\gamma-\delta)$ would vanish identically.

will vanish if any set of three roots are all equal. In like manner, for a quintic there are ten sets of three: $\alpha-\beta$ belongs to three sets, $\gamma-\delta$ to three other sets; the remaining sets are $\alpha\gamma\varepsilon$, $\alpha\delta\varepsilon$, $\beta\gamma\varepsilon$, $\beta\delta\varepsilon$, two of which contain $\gamma-\varepsilon$ and the other two $\delta-\varepsilon$. The function then $\Sigma(\alpha-\beta)^4(\gamma-\delta)^2(\delta-\varepsilon)^2(\gamma-\varepsilon)^2$ is an invariant which will vanish if any set of three roots are all equal. This invariant (Arts. 57, 58) is of the fourth order and its weight is 10.

So, again, if we wish to form a covariant of a biquadratic which shall vanish when two distinct pairs of roots are equal, the expression must contain a difference from each of the pairs $\alpha-\beta$, $\gamma-\delta$; $\alpha-\gamma$, $\beta-\delta$; $\alpha-\delta$, $\beta-\gamma$. Such an expression would be

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)^4,$$

or $$\Sigma(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(x-\beta)^2(x-\gamma)^2(x-\delta)^2,$$

which are covariants of the fourth and sixth degrees respectively in the variables; and of the fourth and third in the coefficients, and every term of each vanishes when two distinct pairs of roots are equal.

139. *Mutual differentiation of covariants and contravariants.* When we say that $\phi(a, b, \xi, \eta,$ &c.) is a contravariant, ξ, η, &c. may be any quantities which are supposed to be transformed by the reciprocal substitution. Now we have shewn (Art. 130) that the differential symbols $\frac{d}{dx}$, $\frac{d}{dy}$, &c. are so transformed. We may, therefore, in any contravariant substitute these differential symbols for ξ, η, &c., and we shall obtain an operating symbol unaltered by transformation, and which, therefore, if applied either to the quantic itself or to any of its covariants, will give a covariant if any of the variables remain after differentiation; and if not, an invariant. Similarly, if applied to a mixed concomitant, it will give either a contravariant or a new mixed concomitant, according as the variables are or are not removed by differentiation. Or, again, in any contravariant instead of obtaining an operating symbol by substituting for ξ, η, &c., $\frac{d}{dx}$, $\frac{d}{dy}$, &c., we may substitute $\frac{dU}{dx}$, $\frac{dU}{dy}$, &c. where

U is either the quantic itself or any of its covariants, and so obtain a new covariant. The relation between the sets of variables x, y, z, &c., ξ, η, ζ, &c. being reciprocal, we may, in like manner, substitute in any covariant, for x, y, z, &c., $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c., when we get an operative symbol which when applied to any contravariant will give either a new contravariant or an invariant.

Thus then, if we are given any covariant and contravariant, by substituting in one of them differential symbols and operating on the other, we obtain a new contravariant or covariant; which again may be combined with one of the two given at first, so as to generate another; and so on.

140. In the case of a binary quantic, this method may be stated more simply. The formulæ for direct transformation being

$$x = \lambda_1 X + \mu_1 Y, \quad y = \lambda_2 X + \mu_2 Y,$$

those for the reciprocal transformation are (Art. 129)

$$\Xi = \lambda_1 \xi + \lambda_2 \eta, \quad \mathrm{H} = \mu_1 \xi + \mu_2 \eta,$$

whence $$\Delta\xi = \mu_2 \Xi - \lambda_2 \mathrm{H}, \quad \Delta\eta = -\mu_1 \Xi + \lambda_1 \mathrm{H},$$

which may be written

$$\Delta\eta = \lambda_1 \mathrm{H} + \mu_1 (-\Xi); \quad \Delta(-\xi) = \lambda_2 \mathrm{H} + \mu_2 (-\Xi).$$

Thus we see that, with the exception of the constant factor Δ, η and $-\xi$ are transformed by exactly the same rules as x and y; and it may be said that y and $-x$ are contragredient to x and y. Thus then, in binary quantics, covariants and contravariants are not essentially distinct, and we have only in any covariant to write η and $-\xi$ for x and y, when we have a contravariant, or *vice versâ*. In fact, suppose that by transformation any homogeneous function whatever $\phi(x, y)$ becomes $\Phi(X, Y)$, the formulæ just given shew that $\phi(\eta, -\xi)$ will become $\Delta^{-p}\Phi(\mathrm{H}, -\Xi)$, where p is the degree of the function in x and y. If then $\phi(x, y)$ is a covariant, that is to say, a function which becomes by transformation one differing only by a power of Δ from a function of like form in X and Y, evidently $\phi(\eta, -\xi)$ will by transformation become one differing only by

a power of Δ from one of like form in Ξ and H; that is to say, it will be a contravariant. For example, the contravariant, noticed (Art. 134, Ex. 1), $c\xi^2 - 2b\xi\eta + a\eta^2$, by the substitution just mentioned, becomes the original quantic.

Instead then of saying that the differential symbols are contragredient to x and y, we may say that they are cogredient to y and $-x$; and if either in the quantic itself or any of its covariants we write $\frac{d}{dy}$, $-\frac{d}{dx}$ for x and y, we get a differential symbol which may be used to generate new covariants in the manner explained in the last article. Or we may substitute $\frac{du}{dy}$, $-\frac{du}{dx}$ for x and y, and so get a new covariant. The following examples will sufficiently illustrate this method:

Ex. 1. To find an invariant of a quadratic, or of a system of two quadratics. Suppose that by transformation $ax^2 + 2bxy + cy^2$ becomes $AX^2 + 2BXY + CY^2$, then since we have seen that $\Delta\frac{d}{dy}$, $-\Delta\frac{d}{dx}$ are transformed by the same rules as x and y, it follows that the operative symbol

$\Delta^2\left(a\frac{d^2}{dy^2} - 2b\frac{d^2}{dx\,dy} + c\frac{d^2}{dx^2}\right)$ becomes by transformation $\left(A\frac{d^2}{dY^2} - 2B\frac{d^2}{dX\,dY} + C\frac{d^2}{dX^2}\right)$.

If then we operate on the given quadratic itself, we get

$$4\Delta^2(ac - b^2) = 4(AC - B^2),$$

which shows that $ac - b^2$ is an invariant; or if we operate on $a'x^2 + 2b'xy + c'y^2$ and the transformed function, we get

$$2\Delta^2(ac' + ca' - 2bb') = 2(AC' + CA' - 2BB'),$$

which shews that $ac' + ca' - 2bb'$ is an invariant. We might also infer that

$$a(bx + cy)^2 - 2b(bx + cy)(ax + by) + c(ax + by)^2$$

is a covariant; but this is only the quantic itself multiplied by $ac - b^2$.

Ex. 2. Every binary quantic of even degree has an invariant of the second order in the coefficients. We have only to substitute, as just explained, $\frac{d}{dy}$, $-\frac{d}{dx}$ for x and y, and operate on the quantic itself. Thus for the quartic $(a, b, c, d, e)(x, y)^4$ we find that $ae - 4bd + 3c^2$ is an invariant; or for the general quantic $(a_0, a_1 \ldots a_{n-1}, a_n)(x, y)^n$, we find that $a_0a_n - na_1a_{n-1} + \frac{1}{2}n(n-1)a_2a_{n-2}$ - &c. is an invariant; where the coefficients are those of the binomial, but the middle term is divided by two.

If we apply this method to a quantic of odd degree; as, for example, if we operate on the cubic $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, with $d\frac{d^3}{dx^3} - 3c\frac{d^3}{dx^2\,dy} + 3b\frac{d^3}{dx\,dy^2} - a\frac{d^3}{dy^3}$, it will be found that the result vanishes identically. We thus find, however, that a system of two cubics has the invariant $(ad' - a'd) - 3(bc' - b'c)$. Or, in general, that a system of two quantics of odd degree, a_0x^n + &c., b_0x^n + &c., has the invariant

$$(a_0b_n - a_nb_0) - n(a_1b_{n-1} - a_{n-1}b_1) + \tfrac{1}{2}n(n-1)(a_2b_{n-2} - a_{n-2}b_2) \text{ \&c.},$$

which vanishes when the two quantics are identical.

141. When, by the method just explained, we have found an invariant of a quantic of any degree, we have immediately, by the method of Art. 126, a covariant of any quantic of higher degree. Thus, knowing that $ac - b^2$ is an invariant of a quadratic, by forming that invariant of the quadratic emanant of any quantic, we learn that $\frac{d^2u}{dx^2}\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2$ is a covariant of any quantic above the second degree. In like manner, from the invariant of a quartic $ae - 4bd + 3c^2$, we infer that for every quantic above the fourth degree

$$\frac{d^4u}{dx^4}\frac{d^4u}{dy^4} - 4\frac{d^4u}{dx^3dy}\frac{d^4u}{dx\,dy^3} + 3\left(\frac{d^4u}{dx^2dy^2}\right)^2$$

is a covariant, &c. In this way we see that a quantic in general has a series of covariants, of the second order in the coefficients, and of the orders $2(n-2)$, $2(n-4)$, $2(n-6)$, &c. in the variables. These covariants may be combined with the original quantic and with each other, so as to lead to new covariants or invariants.

Ex. 1. A quartic has an invariant of the third order in the coefficients. We know that its Hessian

$$(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2,$$

or $(ac - b^2)x^4 + 2(ad - bc)x^3y + (ae + 2bd - 3c^2)x^2y^2 + 2(be - cd)xy^3 + (ce - d^2)y^4$,

is a covariant. Operate on this with $\left(a, b, c, d, e)(\frac{d}{dy}, -\frac{d}{dx}\right)^4$, and we get seventy-two times

$$ace + 2bcd - ad^2 - eb^2 - c^3,$$

which is therefore an invariant.

Ex. 2. Every quantic of odd degree has an invariant of the fourth order in the coefficients. The quantic has a quadratic covariant $\frac{d^{n-1}u}{dx^{n-1}}\frac{d^{n-1}u}{dy^{n-1}}$ – &c. of the second order in the coefficients; and the discriminant of this quadratic will be an invariant of the original quantic (Art. 124), and will be of the fourth order in its coefficients. In fact, it is proved in this way that *every* quantic has an invariant of the fourth order; for if we take any of the covariants of this article, which are all of even degree, its invariant of the second order will be of the fourth order in the coefficients of the original quantic. But when the quantic is of even degree, it *may* happen that the invariant so found is only the square of its invariant of the second order.

Ex. 3. To form the invariant of the fourth order for a cubic.

Its Hessian is $(ax + by)(cx + dy) - (bx + cy)^2$;

or $(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2$.

Hence $(ad - bc)^2 - 4(ac - b^2)(bd - c^2)$
is an invariant of the cubic. In fact, it is its discriminant
$$a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2.$$

142. From any invariant of a binary quantic we can generate a covariant. For from it we can form (Art. 134) the evectant contravariant $\xi^n \frac{dI}{da_0} + \&c.$; and then in this substituting $y, -x$ for ξ and η, we have a covariant. For example, from the discriminant of a cubic which has been just written we form the evectant

$$\xi^3(ad^2 - 3bcd + 2c^3) + 3\xi^2\eta(-acd + 2b^2d - bc^2)$$
$$+ 3\xi\eta^2(-abd + 2ac^2 - b^2c) + \eta^3(a^2d - 3abc + 2b^3),$$

whence we infer that the cubic has the cubic covariant

$$(a^2d - 3abc + 2b^3, abd - 2ac^2 + b^2c, -acd + 2b^2d - bc^2, 3bcd - ad^2 - 2c^3)(x,y)^3.$$

143. *The differential equation.*—We saw (Art. 62) that invariants satisfy certain partial differential equations, and these furnish a third method of forming these functions based on the following principle. *If n be the order of a binary quantic, θ the order in the coefficients of any of its invariants, then the weight* (see Art. 56) *of every term in the invariant is constant and $= \frac{1}{2}n\theta$.* For if we alter x into λx, leaving y unchanged, since this is a linear transformation, the invariant must, by definition, remain unaltered, except that it may be multiplied by a power of λ, which is in this case the modulus of transformation. It is proved then, precisely as in Art. 57, that the *weight*, or sum of the suffixes, in every term is constant.

Again, the invariant must remain unaltered, if we change x into y, and y into x, a linear transformation, the modulus of which is -1. The effect of this substitution is the same as if for each coefficient a_α we substitute $a_{n-\alpha}$. Hence the sum of a number of suffixes

$$\alpha + \beta + \gamma + \&c. = (n - \alpha) + (n - \beta) + (n - \gamma) + \&c.,$$

whence $2(\alpha + \beta + \gamma + \&c.) = n\theta$. Q. E. D.

Cor. n and θ cannot both be odd, since their product is an even number; or, *a binary quantic of odd degree cannot have an invariant of odd order.*

144. The principle just established enables us to write down immediately the literal part of any invariant whose order is given. For the order being given, the weight is given also. Thus, if it were required to form for a quartic an invariant of the third order in the coefficients, the weight must be 6, and the terms of the invariant must be

$$Aa_4a_2a_0 + Ba_4a_1a_1 + Ca_3a_3a_0 + Da_3a_2a_1 + Ea_2a_2a_2,$$

where the coefficients A, B, &c. remain to be determined. The reader will observe that there are as many terms in this invariant as the ways in which the number 6 can be expressed as the sum of three numbers from 0 to 4 inclusive; and generally that there may be as many terms in any invariant as the ways in which its weight $\frac{1}{2}n\theta$ can be expressed as the sum of θ numbers from 0 to n inclusive.

We determine the coefficients from the consideration that since an invariant is to be unaltered by the substitution either of $x+\lambda$ for x, or of $y+\lambda$ for y, evidently, as in Art. 62, every invariant must satisfy the two differential equations

$$a_0\frac{dI}{da_1} + 2a_1\frac{dI}{da_2} + 3a_2\frac{dI}{da_3} + \&c. = 0, \quad na_1\frac{dI}{da_0} + (n-1)a_2\frac{dI}{da_1} + \&c. = 0,$$

it being supposed that the original equation has been written with binomial coefficients. In practice only one of these equations need be used; for the second is derived from the first by changing each coefficient a_α into $a_{n-\alpha}$. It is sufficient then to use one of the equations, provided we take care that the function we form is symmetrical with regard to x and y; that is to say, does not change (or at most changes sign)* when we change a_α into $a_{n-\alpha}$. And this condition will always be fulfilled if we take care that the weight of the invariant is that which has been just assigned. Thus then, in the example chosen for an illustration, if we operate on $Aa_4a_2a_0 + \&c.$ with

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* When we change x into y and y into x, this is a transformation whose modulus is $\begin{vmatrix} 0, & 1 \\ 1, & 0 \end{vmatrix}$ or -1. Any invariant, therefore, which when transformed becomes multiplied by an odd power of the modulus of transformation will change sign when we interchange x and y. Such invariants are called *skew* invariants.

$a_0 \frac{d}{da_1} + \&c.$, we get

$$(2B+2A)\, a_4a_1a_0 + (D+6C+4A)\, a_3a_2a_0 + (2D+4B)\, a_3a_1a_1 + (6E+3D)\, a_2a_2a_1 = 0,$$

whence if we take $A=1$, the other coefficients are found to be $B=-1$, $D=2$, $C=-1$, $E=-1$, and the invariant is

$$a_4a_2a_0 + 2a_3a_2a_1 - a_4a_1a_1 - a_3a_3a_0 - a_2a_2a_2.$$

145. In seeking to determine an invariant of given order by the method just explained, we have a certain number of unknown coefficients A, B, C, &c. to determine, and we do so by the help of a certain number of conditions formed by means of the differential equation. Now, evidently, if the number of these conditions were greater than the number of unknown coefficients, the formation of the invariant would in general be impossible; if they were equal we could form one invariant; if the number of conditions were less, we could form more than one invariant of the given order. We have just seen that the number of terms in the invariant, which is one more than the number of unknown coefficients, is equal to the number of ways in which its weight $\frac{1}{2}n\theta$ can be written, as the sum of θ numbers, none being greater than n. But the effect of the operation $a_0 \frac{d}{da_1} + \&c.$ is evidently to diminish the weight by one, the number of conditions to be fulfilled is, therefore, equal to the number of ways in which $\frac{1}{2}n\theta - 1$ can be expressed as the sum of θ numbers, none exceeding n. Thus, in the example of Art. 144, the number of conditions used to determine A, B, &c. was equal to the number of ways in which 5 can be expressed as the sum of three numbers from 0 to 4 inclusive. To find then generally whether an invariant of a binary quantic of the order θ can be formed, and whether there can be more than one, we must compare the number of ways in which the numbers $\frac{1}{2}n\theta$, $\frac{1}{2}n\theta - 1$ can be expressed as the sum of θ numbers from 0 to n inclusive.*

* It was in this way Prof. Cayley first attempted to investigate the number of invariants and covariants of a binary quantic.

146. Similar reasoning applies to covariants. A covariant, like the original quantic, must remain unaltered, when we change x into ρx, and at the same time every coefficient a_α into $\rho^\alpha a_\alpha$. If then the coefficient of any power of x, x^μ in the covariant be $a\, b_\beta c_\gamma$, &c., it is obvious, as before, that $\mu+\alpha+\beta+$ &c. must be constant for every term; and we may call this number the *weight* of the covariant.

Again, in order that the covariant may not change when we alter x into y and y into x, we must have

$$\mu+\alpha+\beta+\gamma+\text{&c.}=(p-\mu)+(n-\alpha)+(n-\beta)+\text{&c.},$$

where p is the degree of the covariant in x and y; whence if θ be the order of the covariant in the coefficients, we have immediately its weight $=\frac{1}{2}(n\theta+p)$. Thus if it were required to form a quadratic covariant to a cubic, of the second order in the coefficients, $n=3$, $\theta=p=2$, and the weight is 4. We have then for the terms multiplying x^2, $\alpha+\beta=2$, and these terms must be a_2a_0 and a_1a_1. In like manner the terms multiplying xy must be a_3a_0, a_2a_1, and those multiplying y^2 must be a_3a_1, a_2a_2. In this manner we can determine the literal part of a covariant of any order. The coefficients are determined as follows:

147. From the definition of a covariant it follows that we must get the same result whether in it we change x into $x+\lambda y$, or whether we make the same change in the original quantic and then form the covariant. But this change in the original quantic is equivalent (Art. 62) to changing a_1 into $a_1+a_0\lambda$, a_2 into $a_2+2a_1\lambda+a_0\lambda^2$, &c. Hence, in the covariant also the change of x to $x+\lambda y$ must be equivalent to changing a_1 into $a_1+a_0\lambda$, &c. Let the covariant then be

$$A_0x^p+pA_1x^{p-1}y+\tfrac{1}{2}p\,(p-1)\,A_2x^{p-2}y^2+\text{&c.}$$

Let us express that these two alterations are equivalent, and let us confine our attention to the terms multiplying λ. Then if, as in Art. 64, we use as an abbreviation to denote the operation $a_0\dfrac{d}{da_1}+2a_1\dfrac{d}{da_2}+$ &c., the symbol $\dfrac{d}{d\zeta}$, we get

$$\frac{dA_0}{d\zeta}=0,\ \frac{dA_1}{d\zeta}=A_0,\ \frac{dA_2}{d\zeta}=2A_1,\text{&c.},\ \frac{dA_{p-1}}{d\zeta}=(p-1)A_{p-2},\ \frac{dA_p}{d\zeta}=pA_{p-1}.$$

In like manner, writing $\frac{d}{d\eta}$ for $na_1\frac{d}{da_0}+(n-1)a_2\frac{d}{da_1}+$&c., we have

$$\frac{dA_0}{d\eta}=pA_1,\ \frac{dA_1}{d\eta}=(p-1)A_2,\ \&c.$$

Thus we see that when A_p is determined so as to satisfy $\frac{dA_0}{d\zeta}=0$; in other words, when A_0 is a function of the differences of the roots of the quantic (Art. 58), all the other terms of the covariant are known. The covariant is in fact

$$A_0x^p+\frac{dA_0}{d\eta}x^{p-1}y+\frac{d^2A_0}{d\eta^2}\frac{x^{p-2}y^2}{1.2}+\frac{d^3A_0}{d\eta^3}\frac{x^{p-3}y^3}{1.2.3}+\&c.$$

It will be observed that the weight of the covariant being $\frac{1}{2}(n\theta+p)$ the weight of the term A_0 is $\frac{1}{2}(n\theta-p)$, since the weight of A_0 together with p makes up the weight of the covariant. This term A_0, whence all the other terms are derived, was named by Prof. M. Roberts the *source* of the covariant. He observed also that the source of the product of two covariants is the product of their sources. For if we multiply the covariant last written by

$$B_0x^q+\frac{dB_0}{d\eta}x^{q-1}y+\frac{d^2B_0}{d\eta^2}\frac{x^{q-2}y^2}{1.2}+\&c.,$$

we get, as may be easily seen,

$$A_0B_0x^{p+q}+\frac{d(A_0B_0)}{d\eta}x^{p+q-1}y+\frac{d^2(A_0B_0)}{d\eta^2}\frac{x^{p+q-2}y^2}{1.2}+\&c.$$

Hence, if we know any relation connecting any functions of the differences A_0, B_0, C_0, &c., the same relation will connect the covariants derived from these functions.

Ex. 1. To find the quadratic covariant of a cubic. We have seen (Art. 146) that A_0 is of the form $a_2a_0+Ba_1a_1$. Operate on this with $a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}$, and we get $(2+2B)\,a_0a_1=0$, whence $B=-1$ and $A_0=a_2a_0-a_1a_1$. Operate then with $3a_1\frac{d}{da_0}+2a_2\frac{d}{da_1}+a_3\frac{d}{da_2}$, and we have $2A_1=a_3a_0-a_2a_1$. Operate with the same on A_1, and we have $A_2=a_1a_3-a_2a_2$. The covariant, therefore, is

$$(a_2a_0-a_1a_1)\,x^2+(a_0a_3-a_2a_1)\,xy+(a_1a_3-a_2a_2)\,y^2.$$

Ex. 2. To find a cubic covariant of a cubic of the third order in the coefficients. Here $n = 3$, $\theta = 3$, $\frac{1}{2}(n\theta + p) = 6$. The sum then of the suffixes of the coefficient of x^3 will be 3; and this coefficient must be of the form $Aa_3a_0a_0 + Ba_2a_1a_0 + Ca_1a_1a_1$. Operate with $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3}$, and we get

$$(3A + B)\, a_2a_0a_0 + (2B + 3C)\, a_1a_1a_0,$$

whence if we take $A = 1$, we have $B = -3$, $C = 2$, or $A_0 = a_3a_0a_0 - 3a_2a_1a_0 + 2a_1a_1a_1$. Operate on this three times successively with $3a_1 \frac{d}{da_0} + 2a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2}$, and we have the remaining coefficients, and the covariant is (see Art. 142)

$$(a_3a_0a_0 - 3a_2a_1a_0 + 2a_1a_1a_1)\, x^3 + 3\,(a_3a_1a_0 - 2a_2a_2a_0 + a_2a_1a_1)\, x^2y$$
$$+ 3\,(2a_3a_1a_1 - a_2a_2a_1 - a_3a_2a_0)\, xy^2 + (3a_3a_2a_1 - 2a_2a_2a_2 - a_3a_3a_0)\, y^3.$$

148. We have seen that a quantic has as many covariants of the degree p in the variables and of the order θ in the coefficients as functions A_0, whose weight is $\frac{1}{2}(n\theta - p)$ can be found to satisfy the equation $\frac{dA_0}{d\zeta} = 0$. And, as in Art. 145, we see that this number is equal to the difference of the ways in which the numbers $\frac{1}{2}(n\theta - p)$ and $\frac{1}{2}(n\theta - p) - 1$ can be expressed as the sum of θ numbers from 0 to n inclusive. It may be remarked that p cannot be odd unless both n and θ are odd. Hence only quantics of odd degree can have covariants of odd ~~order in the coefficients~~ degree in the variables, and these must also be of odd degree in the variables.

149. The results arrived at (Art. 147) may be stated a little differently. The operation $y \frac{d}{dx}$ performed on any quantic is equivalent to a certain operation performed by differentiating with respect to the coefficients. Thus, for the quantic $(a_0, a_1, a_2 \ldots)(x, y)^n$, we get the same result whether we operate on it with $y \frac{d}{dx}$ or with $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.$ This latter operation then may be written $\left[y \frac{d}{dx}\right]$; and the property already proved for a covariant may be written that we have for it $y \frac{d}{dx} - \left[y \frac{d}{dx}\right] = 0$. In other words, that we get the same

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result whether we operate on the covariant with $y\frac{d}{dx}$ or with $a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+$ &c. In his Memoirs on Quantics, Prof. Cayley has started with this property as his *definition* of a covariant; a definition which includes invariants also, since for them we have $y\frac{dI}{dx}=0$, and therefore also $\left[y\frac{dI}{dx}\right]=0$.

150. It can be proved, in like manner, that covariants of quantics in any number of variables satisfy differential equations which may be written $y\frac{d}{dx}=\left[y\frac{d}{dx}\right]$, $z\frac{d}{dx}=\left[z\frac{d}{dx}\right]$, &c. Thus, for the quantic $(a, b, c, f, g, h)(x, y, z)^2$, we have

$$y\frac{d}{dx}=a\frac{d}{dh}+g\frac{d}{df}+2h\frac{d}{db},\quad z\frac{d}{dx}=a\frac{d}{dg}+h\frac{d}{df}+2g\frac{d}{dc},$$

and every covariant must satisfy these two equations. While every invariant must satisfy the two equations

$$a\frac{dI}{dh}+g\frac{dI}{df}+2h\frac{dI}{db}=0,\quad a\frac{dI}{dg}+h\frac{dI}{df}+2g\frac{dI}{dc}=0;$$

as may easily be proved from the consideration that the invariant remains unaltered if we substitute for x, $x+\lambda y$ or $x+\mu z$.

LESSON XIV.

SYMBOLICAL REPRESENTATION OF CONCOMITANTS.

151. It remains to explain a *fourth* method of finding invariants and covariants, given by Prof. Cayley in 1846 (*Cambridge and Dublin Mathematical Journal*, vol. I. p. 104, and *Crelle*, vol. XXX.); which not only enables us to arrive at such functions, but also affords the basis of a regular calculus by means of which they may be compared and identified.

Let $x_1, y_1; x_2, y_2$ be any two cogredient sets of variables; then, if we write briefly for $\frac{d}{dx_1}, \frac{d}{dy_1}, \frac{d}{dx_2}$; ξ_1, η_1, ξ_2, &c., it has been proved (Arts. 130, 120, 139) that $\xi_1, \eta_1, \xi_2, \eta_2$ are transformed by the reciprocal substitution; that $\xi_1\eta_2 - \xi_2\eta_1$ is an invariant symbol of operation; and that if we operate with any power of this symbol on any function of x_1, y_1, x_2, y_2, we obtain a covariant of that function. We shall use for $\xi_1\eta_2 - \xi_2\eta_1$ the abbreviation $\overline{12}$.

Suppose now that we are given any two binary quantics U, V, we can at once form covariants of this system of two quantics. For we have only to write the variables in U with the suffix (1), those in V with the suffix (2), and then operate on the product UV with any power of the symbol $\overline{12}$; the result must be an invariant or covariant. Thus if we operate simply with $\overline{12}$ we obtain the Jacobian $\frac{dU}{dx}\frac{dV}{dy} - \frac{dU}{dy}\frac{dV}{dx}$, which we saw (Art. 128) was a covariant of the system of quantics.

Again, let

$$U = ax_1^2 + 2bx_1y_1 + cy_1^2; \quad V = a'x_2^2 + 2b'x_2y_2 + c'y_2^2,$$

then if we operate on UV with $\overline{12}^2$, which, written at full length, is

$$\xi_1^2\eta_2^2 + \xi_2^2\eta_1^2 - 2\xi_1\eta_1\xi_2\eta_2,$$

the result is $ac' + ca' - 2bb'$, which is thus proved to be an invariant of this system of quantics. In general, it is obvious

that the differentials marked with the suffix (1) only apply to U, and those with the suffix (2) only to V; and it is unnecessary to retain the suffixes after differentiation;* so that $\overline{12}^2$ applied to two quantics of any degree gives the covariant

$$\frac{d^2U}{dx^2}\frac{d^2V}{dy^2} + \frac{d^2U}{dy^2}\frac{d^2V}{dx^2} - 2\frac{d^2U}{dx\,dy}\frac{d^2V}{dx\,dy}.$$

Similarly the symbol $\overline{12}^3$ applied to two cubics gives the invariant

$$(ad' - da') - 3(bc' - cb'),$$

or to any two quantics gives the covariant

$$\frac{d^3U}{dx^3}\frac{d^3V}{dy^3} - 3\frac{d^3U}{dx^2dy}\frac{d^3V}{dx\,dy^2} + 3\frac{d^3U}{dx\,dy^2}\frac{d^3V}{dx^2dy} - \frac{d^3U}{dy^3}\frac{d^3V}{dx^3};$$

and so in like manner for the other powers of $\overline{12}$.

152. We can by this method obtain also invariants or covariants of a single function U. It is, in fact, only necessary to suppose in the last Article the quantics U and V to be identical. Thus, for instance, in the example of the two quadratics given in the last Article, if we make $a = a'$, $b = b'$, $c = c'$, the invariant $\overline{12}^2$ becomes $2(ac - b^2)$. And, in like manner, the expression there given for the covariant $\overline{12}^2$ of a system U, V, by making $U = V$, gives the covariant of a single quantic

$$\frac{d^2U}{dx^2}\frac{d^2U}{dy^2} - \left(\frac{d^2U}{dx\,dy}\right)^2.$$

In general, whenever we want by this method to form the covariants of a single function, we resort to this process:—We first form a covariant of a system of distinct quantics, and then suppose the quantics to be made identical. And in what follows, when we use such symbols as $\overline{12}^n$ &c. without adding any subject of operation, we mean to express derivatives of a

* If W be any function containing $x_1, y_1; x_2, y_2$; we get the same result whether we linearly transform these variables, and afterwards omit all the suffixes in the transformed equation; or whether we omit the suffixes first, and afterwards transform x and y. This results immediately from the fact that $x_1, y_1; x_2, y_2; x, y$ are cogredient. It follows then at once that if W, written as a function of $x_1, y_1; x_2, y_2$, be a covariant of U, V; that is to say, if the expression of the coefficients of W in terms of the coefficients of U and V be unaffected by transformation, then W is also a covariant when the suffixes are all omitted.

single function U. We take for the subject operated on the product of two or more quantics U_1, U_2, &c., where the variables x_1, y_1; x_2, y_2; &c. are written in each respectively, instead of x and y; and we suppose that after differentiation all the suffixes are omitted, and that the variables, if any remain, are all made equal to x and y.

153. From the omission of the suffixes after differentiation, it follows at once that it cannot make any difference what figures had been originally used, and that $\overline{12}^n$ and $\overline{34}^n$ are expressions for the same thing. In the use of this method we have constantly to employ transformations depending on this obvious principle. Thus, we can show that when n is odd, $\overline{12}^n$ applied to a single function vanishes identically. For, from what has been said, $\overline{12}^n = \overline{21}^n$; but $\overline{12}$ and $\overline{21}$ have opposite signs, as appears immediately on writing at full length the symbol for which $\overline{12}$ is an abbreviation. It follows then that $\overline{12}^n$ must vanish when n is odd. Thus, in the expansion of $\overline{12}^3$, given at the end of Art. 151, if we make $U = V$, it will obviously vanish identically. The series $\overline{12}^2$, $\overline{12}^4$, $\overline{12}^6$, &c. gives the series of invariants and covariants which we have already found (Art. 141). It is easy to see that, when n is even, $\overline{12}^n$ applied to $(a_0, a_1, a_2 \ldots \chi x, y)^n$ gives

$$a_0 a_n - n a_1 a_{n-1} + \tfrac{1}{2} n (n-1) a_2 a_{n-2} - \&c.,$$

where the last coefficients must be divided by two, as is evident from the manner of formation. In particular, we thus have the invariants, for the quadratic, $ac - b^2$; for the quartic, $ae - 4bd + 3c^2$; for the sextic, $ag - 6bf + 15ce - 10d^2$; and so on.

154. The results of the preceding Articles naturally extend to any number of functions. We may take any number of quantics U, V, W, &c., and, writing the variables in the first with the suffix (1), those in the second with the suffix (2), in the third with the suffix (3), and so on, we may operate on their product with the product of any number of symbols $\overline{12}^\alpha$, $\overline{23}^\beta$, $\overline{31}^\gamma$, $\overline{14}^\delta$, &c., where, as before, $\overline{23}$ is an abbreviation for $\xi_2\eta_3 - \xi_3\eta_2$, &c. After the differentiation we suppress the suffixes, and we thus get a covariant of the given system of

quantics, which will be an invariant if it happens that no power of x and y appear after differentiation. Any number of the quantics U, V, W, &c., may be identical; and in the case with which we shall be most frequently concerned, viz., where we wish to form derivatives of a single quantic, the subject operated on is $U_1U_2U_3$ &c., where U_1 and U_2 only differ by having the variables written with different suffixes.

It is evident that in this method the order of the derivative in the coefficients will be always equal to the number of different figures in the symbol for the derivative. For if all the functions were distinct, the derivative would evidently contain a coefficient from every one of the quantics U, V, W, &c.; and it will be still true, when U, V, W are supposed identical, that the degree in the coefficients is equal to the number of factors in the product $U_1U_2U_3$ &c., which we operate on. Thus the derivatives considered in the last Article being all of the form $\overline{12}^r$ are of the second order only in the coefficients.

Again, if it were required to find the degree of the derivative in x and y. Suppose, in the first place, that the quantics were distinct, U being of the degree n, V of the degree n', W of the degree n'', and so on; and suppose that in the operating symbol the figure 1 occurs α times; 2, β times; and so on; then, since U is differentiated α times, V, β times, &c., the result is of the degree $(n-\alpha)+(n'-\beta)+(n''-\gamma)+$&c. When the quantics are identical, if there are p factors in the product $U_1U_2\ldots U_p$, which we operate on, the degree of the result in x and y will be $np-(\alpha+\beta+\gamma+$&c.). While again, if there be r factors such as $\overline{12}$ in the operating symbol, it is obvious that $(\alpha+\beta+\gamma+$&c.$)=2r$. It is clear that if we wish to obtain an *in*variant, we must have $\alpha=\beta=\gamma=n$.

155. To illustrate the above principles, we make an examination of all possible invariants of the third order in the coefficients. Since the symbol for these can only contain three figures, its most general form is $\overline{12}^\alpha.\overline{23}^\beta.\overline{31}^\gamma$; while, in order that it should yield an *in*variant, we must have

$$\alpha+\gamma=\alpha+\beta=\beta+\gamma=n,$$

whence $\alpha=\beta=\gamma$. The general form, then, that we have to

examine is $(\overline{12}.\overline{23}.\overline{31})^\alpha$. Again, if α be odd, this derivative vanishes identically; for, as in Art. 153, by interchanging the figures 1 and 2, we have $(\overline{12}.\overline{23}.\overline{31})^\alpha = (\overline{21}.\overline{13}.\overline{32})^\alpha$; but these have opposite signs. It follows, then, that all invariants of the third order are included in the formula $(\overline{12}.\overline{23}.\overline{31})^\alpha$, where α is even. Thus, $\overline{12}^2.\overline{23}^2.\overline{31}^2$ is an invariant of a quartic, since the differentials rise to the fourth degree; $\overline{12}^4.\overline{23}^4.\overline{31}^4$ is an invariant of an octavic; $\overline{12}^6.\overline{23}^6.\overline{31}^6$ of a quantic of the twelfth degree, and so on; only quantics whose degree is of the form $4m$ having invariants of the third order in the coefficients. If we wish actually to calculate one of these, suppose $\overline{12}^2.\overline{23}^2.\overline{31}^2$, write, for brevity, ξ_1, η_1, &c., instead of $\frac{d}{dx_1}$, $\frac{d}{dy_1}$, &c., and we have actually to multiply out

$$(\xi_1\eta_2 - \xi_2\eta_1)^2\,(\xi_2\eta_3 - \xi_3\eta_2)^2\,(\xi_3\eta_1 - \xi_1\eta_3)^2.$$

In the result omit all the suffixes, and replace ξ^4 by $\frac{d^4U}{dx^4}$ &c.; or, when we operate on a quartic, by a_0 the coefficient of x^4, &c. There are many ways which a little practice suggests for abridging the work of this expansion, but we do not think it worth while to give up the space necessary to explain them; and we merely give the results of the expansion of the three invariants just referred to. $\overline{12}^2.\overline{23}^2.\overline{31}^2$ yields the invariant of a quartic already obtained (Art. 141, Ex. 1, and Art. 144), viz.:—

$$a_4a_2a_0 + 2a_3a_2a_1 - a_4a_1^2 - a_0a_3^2 - a_2^3.$$

$\overline{12}^4.\overline{23}^4.\overline{31}^4$ gives

$$a_8(a_4a_0 - 4a_3a_1 + 3a_2a_2) + a_7(-4a_5a_0 + 12a_4a_1 - 8a_3a_2)$$
$$+ a_6(3a_6a_0 - 8a_5a_1 - 22a_4a_2 + 24a_3a_3) + a_5(24a_5a_2 - 36a_4a_3) + 15a_4a_4a_4.$$

And $\overline{12}^6.\overline{23}^6.\overline{31}^6$ gives

$$a_{12}(a_6a_0 - 6a_5a_1 + 15a_4a_2 - 10a_3a_3) + a_{11}(-6a_7a_0 + 30a_6a_1 - 54a_5a_2 + 30a_4a_3)$$
$$+ a_{10}(15a_8a_0 - 54a_7a_1 + 24a_6a_2 + 150a_5a_3 - 135a_4a_4)$$
$$+ a_9(-10a_9a_0 + 30a_8a_1 + 150a_7a_2 - 430a_6a_3 + 270a_5a_4)$$
$$+ a_8(-135a_8a_2 + 270a_7a_3 + 495a_6a_4 - 540a_5a_5)$$
$$+ a_7(-540a_7a_4 + 720a_6a_5) - 280a_6a_6a_6.$$

156. Though the above-mentioned is the only type of invariants of the third order, there is an unlimited number of

covariants, the simplest being $\overline{12}^2.\overline{13}$, which, when expanded, is

$$\frac{d^3U}{dx^3}\frac{d^2U}{dy^2}\frac{dU}{dy} - \frac{d^3U}{dx^2dy}\left(2\frac{d^2U}{dxdy}\frac{dU}{dy} + \frac{d^2U}{dy^2}\frac{dU}{dx}\right)$$
$$+ \frac{d^3U}{dxdy^2}\left(\frac{d^2U}{dx^2}\frac{dU}{dy} + 2\frac{d^2U}{dxdy}\frac{dU}{dx}\right) - \frac{d^3U}{dy^3}\frac{d^2U}{dx^2}\frac{dU}{dx}.$$

When this is applied to a cubic, it gives the evectant obtained already (Art. 142).

The general type of invariants of the fourth order in the coefficients is $(12.34)^\alpha\ (\overline{13}.\overline{24})^\beta\ (\overline{14}.\overline{23})^\gamma$. Thus the discriminant of a cubic is expressed in this notation as $(\overline{12}.\overline{34})^2\ (\overline{13}.\overline{24})$; but we cannot afford space to enter into greater details on this subject.

157. The principles just laid down afford an easy proof of a remarkable theorem first demonstrated by M. Hermite, and to which we shall refer as "Hermite's Law of Reciprocity." *The number of invariants of the n^{th} order in the coefficients possessed by a binary quantic of the p^{th} degree is equal to the number of invariants of the order p in the coefficients possessed by a quantic of the n^{th} degree.* We have already proved that if any symbol $\overline{12}^a.\overline{23}^b.\overline{34}^c$ &c. denotes an invariant of the order p of a quantic of the degree n, then the number of different figures 1, 2, 3, &c., is p, and each figure occurs n times. But we might calculate by the method of Art. 136 an invariant $\Sigma\,(\alpha-\beta)^a\,(\beta-\gamma)^b(\gamma-\delta)^c$ &c., where we replace each symbol $\overline{34}$ by the difference of two roots $(\gamma-\delta)$. This latter is an invariant of a quantic of the p^{th} degree, since there are by hypothesis p roots; and it is of the order n in the coefficients of the equation (Art. 58).

Thus, for example, a quadratic has but the single independent invariant $(\alpha-\beta)^2$, though of course every power of this is also an invariant; and the general type of such invariants is $(\alpha-\beta)^{2m}$. Hence, only quantics of even degree have invariants of the second order in the coefficients, and the general symbol for such invariants is $\overline{12}^{2m}$.

So again, cubics have no invariant except the discriminant $(\alpha-\beta)^2\,(\beta-\gamma)^2\,(\gamma-\alpha)^2$ and its powers; and the discriminant is of the fourth order in the coefficients. Hence, only quantics of

the degree $4m$ have cubic invariants whose general type is $\overline{12}^{2m}.\overline{23}^{2m}.\overline{31}^{2m}$. It will be proved that quartics have two independent invariants, one of the second and one of the third order, in the coefficients; and, of course, any power of one multiplied by any power of the other is an invariant. Hence, quartics have as many invariants of the p^{th} order as the equation $2x+3y=p$ admits of integer solutions; this is, therefore, the number of invariants of the fourth order which a quantic of the p^{th} degree can possess.

158. Hermite has proved that his theorem applies also to covariants of any given degree in x and y; that is to say, that an n^{ic} possesses as many such covariants of the p^{th} order in the coefficients as a p^{ic} has of the n^{th} order in the coefficients. For, consider any symbol, $\overline{12}^{\lambda}.\overline{23}^{\mu}.\overline{34}^{\nu}$ &c., where there are p figures, and the figure 1 occurs a times, 2 occurs b times, and so on; then we have proved that the degree of this covariant in x and y is $(n-a)+(n-b)+$ &c. But we may form the symmetric function

$$\Sigma\,(\alpha-\beta)^{\lambda}\,(\beta-\gamma)^{\mu}\,(\gamma-\delta)^{\nu}\,(x-\alpha)^{n-a}\,(x-\beta)^{n-b}\ \&c.,$$

which has been proved (Art. 137) to be a covariant of the quantic of the p^{th} degree, whose roots are α, β, &c. Every root enters into its expression in the degree n, which is therefore the order of the covariant in the coefficients, and it obviously contains x and y in the same degree as before, viz. $(n-a)+(n-b)+$ &c. Thus, for example, the only covariants which a quadratic has are some power of the quantic itself multiplied by some power of its discriminant, the general type of which is

$$(\alpha-\beta)^{2p}\,(x-\alpha)^{q}\,(x-\beta)^{q},$$

the order of this in the coefficients is $2p+q$, and in x and y is $2q$. Hence we infer that every quantic of the degree $2p+q$ has a covariant of the second order in the coefficients, and of the degree $2q$ in x and y, the general symbol for such covariants being $\overline{12}^{2p}$. When $q=1$, we obtain the theorem given (Art. 141), that every quantic of odd degree has a quadratic covariant.

159. Concomitants of quantics in three or more variables are expressed in a manner similar to that already explained. If

$x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, be cogredient sets of variables, then, by the rule for multiplication of determinants, the determinant

$$x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1)$$

is an invariant, which, by transformation, becomes a similar function multiplied by the modulus of transformation. And if in the above we write for x_1, $\frac{d}{dx_1}$; for y_2, $\frac{d}{dy_2}$; and so on, we obtain an invariantive symbol of operation, which we shall write $\overline{123}$. When, then, we wish to obtain invariants or covariants of any function U, we have only to operate on the product $U_1U_2U_3 \ldots U_p$ with the product of any number of symbols $\overline{123}^\alpha\ \overline{124}^\beta\ \overline{235}^\gamma$ &c, and after differentiation suppress all the suffixes. Thus, for example, let U_1, U_2, U_3 be ternary quadrics, and let the coefficients in U_1 be a, b, c, $2f$, $2g$, $2h$, as at p. 99, then $\overline{123}^2$ expanded is

$$\begin{aligned} &a(b'c'' + b''c' - 2f'f'') + b(c'a'' + c''a' - 2g'g'') + c(a'b'' + a''b' - 2h'h'') \\ &\quad + 2f(g'h'' + g''h' - a'f'' - a''f') + 2g(h'f'' + h''f' - b'g'' - b''g') \\ &\quad + 2h(f'g'' + f''g' - c'h'' - c''h'); \end{aligned}$$

and this when we suppose the three quantics U_1, U_2, U_3, to be identical, or $a = a' = a''$ &c. reduces to six times

$$abc + 2fgh - af^2 - bg^2 - ch^2.$$

If in the above we replace a, the coefficient of x^2, by $\frac{d^2U}{dx^2}$ &c. we get the expansion of $\overline{123}^2$ as applied to any ternary quantic. This covariant is called the Hessian of the quantic.

It is seen, as at Art. 153, that odd powers of the symbol $\overline{123}$ vanish when it is applied to a single quantic. We give as a further example the expansion of $\overline{123}^4$ applied to the quartic,

$$\begin{aligned} &ax^4 + by^4 + cz^4 + 4(a_2x^3y + a_3x^3z + b_3y^3z + b_1y^3x + c_1z^3x + c_2z^3y) \\ &\quad + 6(dy^2z^2 + ez^2x^2 + fx^2y^2) + 12xyz(lx + my + nz). \end{aligned}$$

Then $\overline{123}^4$ is

$$\begin{aligned} &abc - 4(ab_3c_2 + bc_1a_3 + ca_2b_1) + 3(ad^2 + be^2 + cf^2) + 4(a_2b_3c_1 + a_3b_1c_2) \\ &\quad - 12(a_2nd + a_3md + b_1ne + b_3le + c_1mf + c_2lf) \\ &\quad + 12(lb_1c_1 + mc_2a_2 + na_3b_3) + 12(dl^2 + em^2 + fn^2) + 6def - 12lmn. \end{aligned}$$

160. We can express in the same manner functions containing contragredient variables; for if α, β, γ be any variables contragredient to x, y, z, and therefore cogredient with $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$, it follows, as before, that the determinant

$$\alpha\left(\frac{d}{dy_1}\frac{d}{dz_2}-\frac{d}{dy_2}\frac{d}{dz_1}\right)+\beta\left(\frac{d}{dz_1}\frac{d}{dx_2}-\frac{d}{dz_2}\frac{d}{dx_1}\right)+\gamma\left(\frac{d}{dx_1}\frac{d}{dy_2}-\frac{d}{dx_2}\frac{d}{dy_1}\right)$$

(which we shall denote by the abbreviation $\overline{\alpha 12}$) is an invariantive symbol of operation. Thus, if U_1, U_2 be two different quadrics, $\overline{\alpha 12}^2$ is the contravariant called Φ (*Conics*, Art. 377), which expanded is

$$\begin{aligned}&\alpha^2(b'c''+b''c'-2f'f'')+\beta^2(c'a''+c''a'-2g'g'')+\gamma^2(a'b''+a''b'-2h'h'')\\&+2\beta\gamma(g'h''+g''h'-a'f''-a''f')+2\gamma\alpha(h'f''+h''f'-b'g''-b''g')\\&+2\alpha\beta(f'g''+f''g'-c'h''-c''h'),\end{aligned}$$

and which, when the two quadrics are identical, becomes the equation of the polar reciprocal of the quadric.

In like manner, the quantic contravariant to a quartic, which I have called S (*Higher Plane Curves*, p. 78), may be written symbolically $\overline{\alpha 12}^4$, and the quantic T in the same place may be written $\overline{\alpha 12}^2\,\overline{\alpha 23}^2\,\overline{\alpha 31}^2$. In any of these we have only to replace the coefficient of any power of x, x^n by $\frac{d^n}{dx^n}$ &c. to obtain a symbol which will yield a mixed concomitant when applied to a quantic of higher dimensions. Thus $\overline{\alpha 12}^2$ is

$$\alpha^2\left\{\frac{d^2U}{dy^2}\frac{d^2U}{dz^2}-\left(\frac{d^2U}{dy\,dz}\right)^2\right\}+\&c.,$$

which, when applied to a quadric, is a contravariant, but, when applied to a quantic of higher order, contains both x, y, z, as well as the contragredient α, β, γ, and, therefore, is a mixed concomitant.

In general, if we have the symbolical expression for any invariant of a binary quantic, we have only to prefix a contravariant symbol α to every term, when we shall have a contravariant of a ternary quantic of the same order. And in particular it can be proved that if we take the symbolical expression for the discriminant of a binary quantic, and prefix in

this manner a contravariant symbol to each term, we shall have the expression for the polar reciprocal of a ternary quantic.

Thus, the symbol for the discriminant of a binary cubic is $\overline{12}^2.\overline{34}^2.\overline{13}.\overline{24}$, and the polar reciprocal of a ternary cubic is $\overline{\alpha 12}^2.\overline{\alpha 34}^2.\overline{\alpha 13}.\overline{\alpha 24}$, which is obviously of the sixth order in the variables α, β, γ, and of the fourth in the coefficients.

161. If in any contravariant we substitute $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ for α, β, γ, and operate on U, we get a covariant (Art. 139); and the symbol for this covariant is got from that for the contravariant by writing a new figure instead of α. Thus from $\overline{\alpha 23}^2$ is got $\overline{123}^2$, from $\overline{\alpha 23}.\overline{\alpha 24}$ is got $\overline{123}.\overline{124}$, &c. Conversely, if in the symbol for any invariant we replace any figure by a contravariant symbol α, we get the evectant of that invariant. Thus,

$$\overline{123}.\overline{124}.\overline{234}.\overline{314}$$

is an invariant of a cubic, and the evectant of that invariant is

$$\overline{123}.\overline{\alpha 12}.\overline{\alpha 23}.\overline{\alpha 31}.$$

In the case of a binary quantic, this rule assumes a simpler form; for if we substitute a contravariant symbol for 1 in $\overline{12}$, it becomes, when written at full length, $\xi \frac{d}{dy} - \eta \frac{d}{dx}$, but since ξ and η are cogredient with $-y$ and x, this may be written $x \frac{d}{dx} + y \frac{d}{dy}$, and may be suppressed altogether, since it only affects the result with a numerical multiplier. Hence, given the symbol for any invariant of a binary quantic, its evectant is got by omitting all the factors which contain any one figure. Thus,

$$\overline{12}^2.\overline{34}^2.\overline{13}.\overline{24}$$

being the discriminant of a cubic, its evectant, got by omitting the factors which contain 4, is $\overline{12}^2.\overline{13}$.

If in a contravariant of any quantic we substitute $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$ for α, β, γ, we also get a covariant, and the symbol for it is obtained from that for the contravariant by writing a *different*

new figure in place of every α. Thus, from $\overline{\alpha 34}^2$ we get $\overline{134}.\overline{234}$, and so on.

162. In the explanation of symbolical methods which has been hitherto given, I have followed the notation and course of proceeding originally made use of by Prof. Cayley. I wish now to explain some modifications of notation introduced by Aronhold and Clebsch, who have employed these symbolical methods with great success, but who perhaps at first scarcely sufficiently recognized the substantial identity of their methods with those previously given by Prof. Cayley. The variables are denoted x_1, x_2, x_3, &c., while the coefficients are denoted by suffixes corresponding to the variables which they multiply. Thus the ternary cubic, the ternary quartic, &c., may be briefly denoted $\Sigma a_{ikl}x_ix_kx_l$, $\Sigma a_{iklm}x_ix_kx_lx_m$, &c., where the numbers i, k, l, m are to receive in succession all the values 1, 2, 3, &c. It will be observed that in this notation $a_{iik} = a_{iki} = a_{kii}$, so that when we form the sums indicated we obtain a quantic written with the numerical coefficients of the binomial theorem. Thus when we form the sum $\Sigma a_{ikl}x_ix_kx_l$, the three terms $a_{112}x_1x_1x_2$, $a_{121}x_1x_2x_1$, $a_{211}x_2x_1x_1$ are identical, as in like manner are the six terms

$$a_{123}x_1x_2x_3,\ a_{132}x_1x_3x_2,\ a_{213}x_2x_1x_3,\ a_{231}x_2x_3x_1,\ a_{312}x_3x_1x_2,\ a_{321}x_3x_2x_1,$$

so that the sum written at length would be

$$a_{111}x_1x_1x_1 + a_{222}x_2x_2x_2 + a_{333}x_3x_3x_3 + 3a_{112}x_1x_1x_2 + \ldots + 6a_{123}x_1x_2x_3.$$

And so, in like manner, in general. Now Aronhold uses, as an abbreviated expression for the quantic in general,

$$(a_1x_1 + a_2x_2 + a_3x_3 + \ldots)^n,$$

where, after expansion, we are to substitute for the products $a_ia_ka_l$, &c., the coefficients a_{ikl}. Thus the ternary cubic given above may be written in the abbreviated form

$$(a_1x_1 + a_2x_2 + a_3x_3)^3;$$

the terms $a_1a_1a_1x_1x_1x_1 + 3a_1a_1a_2x_1x_1x_2 +$ &c.

in the expansion of the cube being replaced by $a_{111}x_1x_1x_1$, $3a_{112}x_1x_1x_2$, &c. The quantity $a_1x_1 + a_2x_2 + a_3x_3$ is written a_x

or sometimes simply a, and the quantic is symbolically expressed as a_x^n. The cubic might equally have been written $(b_1x_1 + b_2x_2 + b_3x_3)^3$, $(c_1x_1 + c_2x_2 + c_3x_3)^3$, &c., it being understood that we are in like manner to substitute for $b_1b_1b_1$, $c_1c_1c_2$, &c., the coefficients a_{111}, a_{112}, &c. Now the rule given by Aronhold for the formation of invariants is to take a number of determinants, whose constituents are the symbols a_1, a_2, a_3; b_1, b_2, &c., to multiply all together, and after multiplication to substitute for the symbols $a_ia_ka_l$, $b_mb_nb_p$, the coefficients a_{ikl}, a_{mnp}, &c. Thus Aronhold first discovered a fundamental invariant of a ternary cubic by forming the four determinants $\Sigma \pm a_1b_2c_3$, $\Sigma \pm b_1c_2d_3$, $\Sigma \pm c_1d_2a_3$, $\Sigma \pm d_1a_2b_3$; multiplying all together and then performing the substitutions already indicated. This is the same invariant which, in Prof. Cayley's notation, would be designated as $\overline{123}.\overline{234}.\overline{341}.\overline{412}$. In order to obtain an invariant by this method, it is obviously necessary (as in Art. 154) that the a symbols, b symbols, &c. respectively should each occur n times. A product of determinants not fulfilling this condition is made to express a covariant by joining to it such powers of a_x, b_x, &c. as will make up the total number of a's, b's, &c. to n. Thus the Hessian of a binary quadratic, which in Cayley's notation is $\overline{12}^2$ is in Aronhold's $(ab)^2$; but the Hessian of any other binary quantic, which in Cayley's notation is still $\overline{12}^2$, is in Aronhold's $(ab)^2 a_x^{n-2} b_x^{n-2}$.

163. In order to see the substantial identity of the two methods, it is sufficient to observe that by the theorem of homogeneous functions any quantic u of the n^{th} order differs only by a numerical multiplier from $\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3}\right)^n u$, so that if we write it $(a_1x_1 + a_2x_2 + a_3x_3)^n$, the symbols a_1, a_2, a_3 differ only by a numerical constant from the differential symbols $\frac{d}{dx_1}$, &c. And we evidently get the same results whether with Prof. Cayley we form determinants whose constituents are $\frac{d}{dx_1}$, $\frac{d}{dx_2}$, $\frac{d}{dx_3}$, or with Aronhold, whose constituents are a_1, a_2, a_3.

And the artifice made use of by both is the same. If we multiply together a number of differential symbols $\left(\frac{d}{dx}+\lambda\frac{d}{dy}\right)\left(\frac{d}{dx}+\mu\frac{d}{dy}\right)$, &c., and operate on U, it is evident the result will be a linear function of differentials of U of an order equal to the number of factors multiplied together; and that in this way we can never get any power higher than the first of any differential coefficient. When, then, it is required to express symbolically a function involving powers of the differential coefficients, the artifice used by Prof. Cayley was to write the function first with different sets of variables, and form such a function as $\left(\frac{d}{dx_1}+\lambda\frac{d}{dy_1}\right)\left(\frac{d}{dx_2}+\mu\frac{d}{dy_2}\right)U_1U_2$, and *after* differentiation to make the variables identical. So in like manner Aronhold in his symbolic multiplication uses different symbols which have the same meaning after the multiplication has been performed. By multiplying together symbols a_i, a_k, a_l, &c., we can only get a term such as a_{ikl} of no higher than the first order in the coefficients. When, then, we want to express symbolically functions of the coefficients of higher order than the first, the artifice is used of multiplying together different sets of symbols a_i, a_k, a_l; b_i, b_k, b_l, &c., the products $a_ia_ka_l$, $b_ib_kb_l$, $c_ic_kc_l$, &c., all equally denoting the coefficient a_{ikl}.

The notations explained in this Lesson afford a complete calculus, by means of which invariants and covariants can be transformed and the identity of different expressions ascertained. We shall in a subsequent Lesson give illustrations of the applications of this method, referring those desirous of further information to Clebsch's valuable *Theorie der binären algebraischen Formen*, in which work this symbolical method is the basis of the whole treatment of the subject.

LESSON XV.

CANONICAL FORMS.

164. SINCE invariants and covariants retain their relations to each other, no matter how the quantic is linearly transformed, it is plain that when we wish to study these relations it is sufficient to do so by discussing the quantic in the simplest form to which it is possible to reduce it. This is only extending to quantics in general what the reader is familiar with in the case of ternary and quaternary quantics; since, when we wish to study the properties of a curve or surface, every geometer is familiar with the advantage of choosing such axes as shall reduce the equation of this curve or surface to its simplest form.* The simplest form then, to which a quantic can without loss of generality be reduced, is called the *canonical form* of the quantic. We can, by merely counting the constants, ascertain whether any proposed simple form is sufficiently general to be taken as the canonical form of a quantic, for if the proposed form does not, either explicitly or implicitly, contain as many constants as the given quantic in its most general form, it will not be possible always to reduce the general to the proposed form.†

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* It must be owned, however, that as in the progress of analysis greater facility is gained in dealing with quantics in their most general form, the advantage diminishes of reducing them to simpler forms.

† It is not true, however, conversely, that a form which contains the proper number of constants is necessarily one to which the general equation may be reduced. For when we endeavour by comparison of coefficients to identify such a form with the general equation, although the number of equations is equal to the number of quantities to be determined, it *may* happen that the constants enter into the equations in such a way that all the equations cannot be satisfied. Thus

$$(x-\alpha)^2+(y-\beta)^2=lx+my+n$$

is a form containing five constants, and yet is not one to which the general equation of a ternary quadric can be reduced; since the constants enter the equation in such a way that though we have more than enough to make the coefficients of x and y and the absolute term identical with those in any proposed equation, we have no means of identifying the coefficients of x^2, xy and y^2. A more important example is

$$x^4+y^4+z^4+u^4+v^4,$$

Thus, a binary cubic may be reduced to the form X^3+Y^3; for the latter form, being equivalent to $(lx+my)^3+(l'x+m'y)^3$, contains implicitly four constants, and therefore is as general as $(a, b, c, d\mathcal{X}x, y)^3$. So, in like manner, a ternary cubic in its most general form contains ten constants; but the form $X^3+Y^3+Z^3+6mXYZ$ contains also ten constants, since, in addition to the m which appears explicitly, X, Y, Z implicitly involve three constants each. This latter, then, may be taken as the canonical form of a ternary cubic, and, in fact, some of the most important advances that have been made in the theory of curves of the third degree are owing to the use of the equation in this simple and manageable form.

165. The quadratic function $(a, b, c\mathcal{X}x, y)^2$ can be reduced in an infinity of ways to the form x^2+y^2, since the latter form implicitly contains four constants, and the former only three. In like manner the ternary quadric which contains six constants can be reduced in an infinity of ways to the form $x^2+y^2+z^2$, since this last contains implicitly nine constants; and, in general, a quadratic form in any number of variables can be reduced in an infinity of ways to a sum of squares. It is worth observing, however, that though a quadratic form can be reduced in an infinity of ways to a sum of squares, yet the number of positive and negative squares in this sum is fixed. Thus, if a binary quadric can be reduced to the form x^2+y^2, it cannot also be reduced to the form u^2-v^2, since we cannot have x^2+y^2 identical with u^2-v^2, the factors on the one side of the identity being imaginary, and those on the other being real. In like manner, for ternary quadrics we cannot have $x^2+y^2-z^2=u^2+v^2+w^2$, since we should thus have $x^2+y^2=z^2+u^2+v^2+w^2$, or, in other words,

$$x^2+y^2=z^2+(lx+my+nz)^2+(l'x+m'y+n'z)^2+(l''x+m''y+n''z)^2,$$

and if we make x and $y=0$, one side of the identity would

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where z, u, v are linear functions. In the case of a ternary quantic this form contains implicitly fourteen independent constants, and therefore seems to be one to which the quartic in general can be reduced. But Clebsch has shewn that a condition must be fulfilled in order that a quartic should be reducible to this form, namely, the vanishing of a certain invariant. See also *Surfaces*, Note to Art. 235.

vanish, and the other would reduce itself to the sum of four positive squares which could not be $= 0$. And the same argument applies in general.

166. We commence by shewing that, as has been just stated, a cubic may always be reduced to the sum of two cubes. To do this is, in fact, to solve the equation, since when the quantic is brought to the form $X^3 + Y^3$, it can immediately be resolved into its linear factors. Now, if the cubic $(a, b, c, d)(x, y)^3$ become by transformation $(A, B, C, D)(X, Y)^3$, then, since (Art. 126) the Hessian $(ax + by)(cx + dy) - (bx + cy)^2$ is a covariant, it will, by the definition of a covariant, be transformed into a similar function of A, B, C, D, X, Y. That is to say, we must have

$$(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2$$
$$= (AC - B^2)X^2 + (AD - BC)XY + (BD - C^2)Y^2.$$

Now, if in the transformed cubic, B and C vanish, the Hessian takes the form $ADXY$; and we see at once that we are to take for X and Y the two factors into which the Hessian may be broken up. When we have found X and Y, we compare the given cubic with $AX^3 + DY^3$, and determine A and D by comparison of coefficients.

Ex. To reduce $4x^3 + 9x^2 + 18x + 17$ to the form $AX^3 + DY^3$. The Hessian is

$$(4x + 3)(6x + 17) - (3x + 6)^2,$$

or

$$15x^2 + 50x + 15,$$

whose linear factors are $x + 3$, $3x + 1$. Comparing then the given cubic with

$$A(x + 3)^3 + D(3x + 1)^3,$$

we have $A + 27D = 4$, $27A + D = 17$, whence $728D = 91$, $728A = 455$, or A is to D in the ratio of 5 to 1. The given cubic then only differs by a factor (viz. 8) from

$$5(x + 3)^3 + (3x + 1)^3,$$

and it is obvious that the roots of the cubic are given by the equation

$$3x + 1 + (x + 3)\sqrt[3]{(5)} = 0.$$

167. It is evident that every cubic cannot be brought by *real* transformation to the form $AX^3 + DY^3$, for this last form has one real factor and two imaginary; and therefore cannot

be identical with a cubic whose three factors are real. The discriminant of the Hessian

$$4(ac-b^2)(bd-c^2)-(ad-bc)^2$$

is, with sign changed, the same as that of the cubic. When the discriminant of the cubic is positive, the Hessian has two real factors, and the cubic one real factor and two imaginary. When it is negative, the Hessian has two imaginary factors, and the cubic three real. When it vanishes, both Hessian and cubic have two equal factors, and it can be directly verified that the Hessian of X^2Y is X^2.*

It is to be observed, that a quantic of the same degree cannot *always* be reduced to the same canonical form. The impossibility of the reduction indicates some singularity in the form of the quantic. Thus a cubic having a square factor cannot be brought to the form Ax^3+Dy^3: a different canonical form must be adopted, and the most simple one is the form x^2y, to which the cubic in question is obviously at once reducible.

168. In the same manner as a cubic can be brought to the sum of two cubes, so in general any binary quantic of odd degree $(2n-1)$ can be reduced to the sum of n powers of the $(2n-1)^{th}$ degree, a theorem due to Dr. Sylvester. For the number of constants in any binary quantic is always one more than its degree, or, in the present case, $2n$; and we have the same number of constants if we take n terms of the form $(lx+my)^{2n-1}$. The actual transformation is performed by a method which is the generalization of that employed (Art. 166). For simplicity, we only apply it to the fifth degree, but the method is general. The problem then is to determine u, v, w, so that $(a, b, c, d, e, f)(x, y)^5$ may $=u^5+v^5+w^5$. Now we say that if we form the determinant

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix},$$

* In general, when a binary quantic has a square factor, this will also be a square factor in its Hessian, as may be verified at once by forming the Hessian of $x^2\phi$.

the three factors of this cubic will be u, v, w. For let

$$u = lx + my, \quad v = l'x + m'y, \quad w = l''x + m''y;$$

then, differentiating the identity

$$(a, b, c, d, e, f)(x, y)^5 = u^5 + v^5 + w^5$$

four times successively with regard to x, and dividing by 120, we get

$$ax + by = l^4u + l'^4v + l''^4w.$$

Similarly differentiating three times with regard to x, and once with regard to y,

$$bx + cy = l^3mu + l'^3m'v + l''^3m''w;$$

and so on.

The determinant, then, written above, may be put into the form

$$\begin{vmatrix} l^4u + l'^4v + l''^4w, & l^3mu + l'^3m'v + l''^3m''w, & l^2m^2u + l'^2m'^2v + l''^2m''^2w \\ l^3mu + l'^3m'v + l''^3m''w, & l^2m^2u + l'^2m'^2v + l''^2m''^2w, & lm^3u + l'm'^3v + l''m''^3w \\ l^2m^2u + l'^2m'^2v + l''^2m''^2w, & lm^3u + l'm'^3v + l''m''^3w, & m^4u + m'^4v + m''^4w \end{vmatrix}$$

But (Art. 22) this is the product

$$\begin{vmatrix} l^2u, & l'^2v, & l''^2w \\ lmu, & l'm'v, & l''m''w \\ m^2u, & m'^2v, & m''^2w \end{vmatrix} \cdot \begin{vmatrix} l^2, & l'^2, & l''^2 \\ lm, & l'm', & l''m'' \\ m^2, & m'^2, & m''^2 \end{vmatrix},$$

or is $$uvw\,(lm' - l'm)^2\,(l'm'' - l''m')^2\,(l''m - lm'')^2.$$

When, then, the determinant written in the beginning of this Article has been found, by solving a cubic equation, to be the product of the factors $(x + \lambda y)(x + \mu y)(x + \nu y)$, we know that u, v, w can only differ from these by numerical coefficients, and we may put

$$(a, b, c, d, e, f)(x, y)^5 = A(x + \lambda y)^5 + B(x + \mu y)^5 + C(x + \nu y)^5;$$

and then A, B, C are found by solving any of the systems of simple equations got by equating three coefficients on both sides of the above identity.

The determinant used in this Article is a covariant, which is called the *canonizant* of the given quintic.

169. The canonizant may be written in another, and perhaps simpler form, namely,

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix}.$$

This last is the form in which we should have been led to it if we had followed the course that naturally presented itself, and sought directly to determine the six quantities A, B, C, λ, μ, ν, by solving the six equations got on comparison of coefficients of the identity last written in Art. 168. For the development of the solution in this form, to which we cannot afford the necessary space here, we refer to Sylvester's Paper (*Philosophical Magazine*, November, 1851). Meanwhile, the identity of the determinant in this Article with that in the last has been shown by Prof. Cayley as follows. We have, by multiplication of determinants (Art. 22).

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ x, & y, & 0, & 0 \\ 0, & x, & y, & 0 \\ 0, & 0, & x, & y \end{vmatrix}$$

$$= \begin{vmatrix} y^3, & 0, & 0, & 0 \\ 0, & ax+by, & bx+cy, & cx+dy \\ 0, & bx+cy, & cx+dy, & dx+ey \\ 0, & cx+dy, & dx+ey, & ex+fy \end{vmatrix},$$

which, dividing both sides of the equation by y^3, gives the identity required.

170. We have still to mention another way of forming the canonizant. Let this sought covariant be $(A, B, C, D\mathbin{]\![} x, y)^3$, where we want to determine A, B, C, D; then (Art. 140) $(A, B, C, D\mathbin{]\![} \frac{d}{dy}, -\frac{d}{dx})^3$ will also yield a covariant. But if this operation is applied to $(x+\lambda y)^n$ where $x+\lambda y$ is a factor in $(A, B, C, D\mathbin{]\![} x, y)^3$, the result must vanish, since one of the

factors in the operating symbol is $\frac{d}{dy} - \lambda \frac{d}{dx}$. Since, then, the given quantic is by hypothesis the sum of three terms of the form $(x + \lambda y)^5$, the result of applying to the given quantic the operating symbol just written must vanish. Thus, then, we have

$$A(d, e, f\mathcal{X}x, y)^2 - B(c, d, e\mathcal{X}x, y)^2 + C(b, c, d\mathcal{X}x, y)^2 - D(a, b, c\mathcal{X}x, y)^2 = 0,$$

or, equating separately to 0 the coefficients of x^2, xy, y^2, we have

$$Ad - Bc + Cb - Da = 0,$$
$$Ae - Bd + Cc - Db = 0,$$
$$Af - Be + Cd - Dc = 0,$$

whence (Art. 28) A is proportional to the determinant got by suppressing the column A or $\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}$ and so for B, C, D, which values give for the canonizant the form stated in the last Article.

171. We proceed now to quantics of even degree $(2n)$. Since this quantic contains $2n + 1$ terms, if we equate it to a sum of n powers of the degree $2n$, we have one equation more to satisfy than we have constants at our disposal. On the other hand, if we add another $2n^{th}$ power, we have one constant too many, and the quantic can be reduced to this form in an infinity of ways. It is easy, however, to determine the condition that the given quantic should be reducible to the sum of n, $2n^{th}$ powers. Thus, for example, the conditions that a quartic should be reducible to the sum of two fourth powers, and that a sextic should be reducible to the sum of three sixth powers, are respectively the determinants

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0, \qquad \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} = 0,$$

and so on. For, in the case of the quartic, the constituents of the determinant are the several fourth differentials of the quantic, and expressing these in terms of u and v precisely as in Art. 168, it is easy to see, Art. 26, Ex. 5, that the determinant must vanish, when the quartic can be reduced to the form u^4+v^4. Similarly for the rest. This determinant expanded in the case of the quartic is the invariant already noticed (see Art. 141, Ex. 1),

$$ace+2bcd-ad^2-eb^2-c^3.$$

172. When this condition is not fulfilled, the quantic is reduced to the sum of n powers, together with an additional term. Thus, the canonical form for a quartic is naturally taken to be $u^4+v^4+6\lambda u^2v^2$. We shall commence with the reduction of the general quartic to this canonical form; the method which we shall use is not the easiest for this case, but is that which shows most readily how the reduction is to be effected in general. Let the product, then, of u, v, which we seek to determine, be $(A, B, C\mathcal{X}x, y)^2$, and let us operate with $(A, B, C\mathcal{X}\frac{d}{dy}, -\frac{d}{dx})^2$ on both sides of the identity $(a, b, c, d, e\mathcal{X}x, y)^4=u^4+v^4+6\lambda u^2v^2$.

Now, as before, this operation performed on u^4 and on v^4 will vanish, and when performed on $6\lambda u^2v^2$ it will be found to give $12\lambda' uv$, where $\lambda'=2(4AC-B^2)\lambda$. Equating then the coefficients of x^2, xy, and y^2 on both sides, after performing the operation, we get the three equations

$$\begin{aligned} Ac-Bb+Ca &= \lambda' A,\\ Ad-Bc+Cb &= \tfrac{1}{2}\lambda' B,\\ Ae-Bd+Cc &= \lambda' C, \end{aligned}$$

whence eliminating A, B, C, we have to determine λ', the determinant

$$\begin{vmatrix} a, & b, & c-\lambda' \\ b, & c+\tfrac{1}{2}\lambda', & d \\ c-\lambda', & d, & e \end{vmatrix} = 0,$$

which, expanded, is the cubic

$$\lambda'^3 - \lambda' (ae - 4bd + 3c^2) - 2 (ace + 2bcd - ad^2 - eb^2 - c^3) = 0,$$*

the coefficients of which are invariants. Thus, then, we have a striking difference in the reduction of binary quantics to their canonical form, between the cases where the degree is odd and where it is even. In the former case, the reduction is unique, and the system u, v, w, &c. can be determined in but one way. When u is of even degree, however, more systems than one can be found to solve the problem. Thus, in the present instance, a quartic can be reduced in three ways to the canonical form, and if we take for λ' any of the roots of the above cubic, its value substituted in the preceding system of equations enables us to determine A, B, C.

173. If now we proceed to the investigation of the reduction of the quantic $(a_0, a_1, a_2 \ldots)(x, y)^{2n}$, the most natural canonical form to assume would be $u^{2n} + v^{2n} + w^{2n} + \&c. + \lambda u^2 v^2 w^2$ &c., there being n quantities u, v, w, &c. But the actual reduction to this form is attended with difficulties which have not been overcome, except for the cases $n = 2$ and $n = 4$. But the method used in the last Article can be applied if we take for the canonical form $u^{2n} + v^{2n} + \&c. + \lambda Vuvw$ &c., where, if

$$uvw \;\&c. = (A_0, A_1, A_2 \ldots)(x, y)^n,$$

V is a covariant of this latter function such that when $Vuvw$ &c. is operated on by $(A_0, A_1 \ldots)\left(\frac{d}{dy}, -\frac{d}{dx}\right)^n$, the result is proportional to the product uvw &c. Suppose, for the moment, that we had found a function V to fulfil this condition, then, proceeding exactly as in the last Article, and operating with the differential symbol last written on the identity got by equating the quantic to its canonical form, we get the system of equations

$$A_0 a_n - A_1 a_{n-1} + A_2 a_{n-2} - \&c. = \lambda' A_0,$$

$$A_0 a_{n+1} - A_1 a_n + A_2 a_{n-1} - \&c. = \frac{1}{n} \lambda' A_1,$$

$$A_0 a_{n+2} - A_1 a_{n+1} + A_2 a_n - \&c. = \frac{2}{n(n-1)} \lambda' A_2, \;\&c.,$$

* N.B.—The discriminant of this cubic is the same as that of the quartic.

whence, eliminating A_0, A_1, A_2, &c., we get the determinant

$$\begin{vmatrix} a_n - \lambda', & a_{n-1}, & a_{n-2}, & \ldots\ldots\ldots\ldots & a_0 \\ a_{n+1}, & a_n + \frac{1}{n}\lambda', & a_{n-1}, & \ldots\ldots\ldots\ldots & a_1 \\ a_{n+2}, & a_{n+1}, & a_n - \frac{1.2}{n(n-1)}\lambda', & \ldots & a_2 \\ \ldots\ldots\ldots\ldots & & & & \\ a_{2n}, & a_{2n-1}, & a_{2n-2}, & \ldots\ldots\ldots\ldots & a_n \mp \lambda'^* \end{vmatrix};$$

and having found λ' by equating to 0 this determinant expanded (a remarkable equation, all the coefficients of which will be invariants), the equations last written enable us to determine the values of A_0, A_1, &c., corresponding to any of the $n+1$ values of λ'.

174. To apply this to the case of the sextic, the canonical form here is $u^6 + v^6 + w^6 + Vuvw$, where, if uvw be

$$(A_0, A_1, A_2, A_3)(x, y)^3,$$

V is the evectant of the discriminant of this last quantic, and its value is written at full length (Art. 142). Now it will afford an excellent example of the use of canonical forms if we show that in any cubic the result of the operation

$$(a_0, a_1, a_2, a_3)\left(\frac{d}{dy}, -\frac{d}{dx}\right)^3,$$

performed on the product of the cubic and the evectant just mentioned, will be proportional to the cubic itself. For it is sufficient to prove this, for the case when the cubic is reduced to the canonical form $x^3 + y^3$, in which case the evectant will be $x^3 - y^3$, as appears at once by putting $b = c = 0$, and $a = d = 1$ in the value given, Art. 142. The product, then, of cubic and

* The determinant above written may be otherwise obtained as follows. Let x', y' be cogredient to x, y, and let us form the function

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^n U + \lambda\,(xy' - yx')^n,$$

which (Arts. 125, 131) we have proved to be linearly transformed into a function of similar form. Equate to zero the $n+1$ coefficients of the several powers x^n, $x^{n-1}y$, &c., and from these eliminate linearly the $n+1$ quantities x'^n, $x'^{n-1}y'$, &c., and we obtain the determinant in question.

evectant will be $x^6 - y^6$, which, if operated on by $\frac{d^3}{dy^3} - \frac{d^3}{dx^3}$, gives a result manifestly proportional to $x^3 + y^3$. And the theorem now proved being independent of linear transformation, if true for any form of the cubic, is true in general. The canonical form, then, being assumed as above, we proceed exactly as in the last Article, and we solve for λ from the equation

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 - \lambda \\ a_1, & a_2, & a_3 + \frac{1}{3}\lambda, & a_4 \\ a_2, & a_3 - \frac{1}{3}\lambda, & a_4, & a_5 \\ a_3 + \lambda, & a_4, & a_5, & a_6 \end{vmatrix} = 0,$$

which, when expanded, will be found to contain only even powers of λ. If we suppose uvw reduced as above to its canonical form $x^3 + y^3$, the three factors of which are

$$x + y, \quad x + \omega y, \quad x + \omega^2 y,$$

where ω is a cube root of unity, then it is evident from the above that the corresponding canonical form for the sextic is

$$A(x+y)^6 + B(x+\omega y)^6 + C(x+\omega^2 y)^6 + D(x^6 - y^6).$$

It can be proved that if u, v, w be the factors of the cubic, then the factors of the evectant used above are $v - w$, $w - u$, $u - v$, so that the canonical form of the sextic may also be written

$$u^6 + v^6 + w^6 + \lambda uvw(u-v)(v-w)(w-u).$$

175. In the case of the octavic the canonical form is

$$u^8 + v^8 + w^8 + z^8 + \lambda u^2 v^2 w^2 z^2,$$

for if we operate on $u^2v^2w^2z^2$ with a symbol formed according to the same method as in the preceding Articles, the result will be proportional to $uvwz$.

As for higher canonical forms we content ourselves with again mentioning that for a ternary cubic, viz. $x^3 + y^3 + z^3 + 6mxyz$, and that given by Sylvester for a quaternary cubic,

$$x^3 + y^3 + z^3 + u^3 + v^3.$$

LESSON XVI.

SYSTEMS OF QUANTICS.

176. It still remains to explain a few properties of systems of quantics, to which we devote this Lesson. An invariant of a system of quantics of the same degree is called a *combinant* if it is unaltered (except by a constant multiplier) not only when the variables are linearly transformed, but also when for any of the quantics is substituted a linear function of the quantics. Thus the eliminant of a system of quantics u, v, w is a combinant. For, evidently the result of substituting the common roots of vw in $u + \lambda v + \mu w$ is the same as that of substituting them in u; and the eliminant of $u + \lambda v + \mu w$, v, w is the same as the eliminant of uvw. In addition to the differential equations satisfied by ordinary invariants, combinants must evidently also satisfy the equation

$$\frac{a'dI}{da} + \frac{b'dI}{db} + \frac{c'dI}{dc} + \&c. = 0.$$

It follows from this that in the case of two quantics a combinant is a function of the determinants (ab'), (ac'), (bd'), &c.; in the case of three, of the determinants $(ab'c'')$, &c.; and will accordingly vanish identically, if any two of the quantics become identical. If we substitute for u, v; $\lambda u + \mu v$, $\lambda' u + \mu' v$, every one of the determinants (ab') will be multiplied by $(\lambda\mu' - \lambda'\mu)$; and therefore the combinant will be multiplied by a power of $(\lambda\mu' - \lambda'\mu)$ equal to the order of the combinant in the coefficients of any of the quantics. Similarly for any number of quantics. There may be in like manner combinantive covariants, which are equally covariants when for any of the quantics is substituted a linear function of them. For instance, the Jacobian (Art. 88)

$$\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ w_1, & w_2, & w_3 \end{vmatrix},$$

if we substitute for u, $lu+mv+nw$, for v, $l'u+m'v+n'w$, &c. by the property of determinants, becomes the product of the determinants $(lm'n'')$, $(u_1v_2w_3)$. The coefficients of a combinantive covariant are also functions of the determinants (ab'), (ac'); $(ab'c'')$, &c.

177. If $u=(a, b, c\ldots\chi x, y)^n$, $v=(a', b', c'\ldots\chi x, y)^n$ be any two binary quantics of the *same* degree, then $u+kv$ or $(a+ka', b+kb'\ldots\chi x, y)^n$, where we give different values to k, denotes a system of quantics which are said to form with u, v an *involution*. Now there will be in general $2(n-1)$ quantics of the system, each of which will have a square factor. For the discriminant of a quantic of the n^{th} degree is of the order $2(n-1)$ in the coefficients (Art. 105). If then we substitute $a+ka'$ for a, $b+kb'$ for b, &c., there will evidently be $2(n-1)$ values of k, for which the discriminant will vanish.

If we make $y=1$ in any of the quantics, it denotes n points on the axis of x. We have just proved that in $2(n-1)$ cases, two of the n points denoted by $u+kv$ will coincide; or, in other words we may say, that there are $2(n-1)$ *double points* in the involution.

When $u+kv$ has a square factor $x-\alpha$, we know that α satisfies the two equations got by differentiation, viz. $u_1+kv_1=0$, $u_2+kv_2=0$, and therefore will satisfy the equation got by eliminating k between them, viz. $u_1v_2-u_2v_1=0$. Now $u_1v_2-u_2v_1$, which is of the degree $2(n-1)$, is the Jacobian of u, v; and we see that by equating the Jacobian to 0, we obtain the $2(n-1)$ double points of the involution determined by u, v.*

178. *If u and v have a common factor, this will appear as a square factor in their Jacobian.* First, let it be observed, that since $nu=xu_1+yu_2$, $nv=xv_1+yv_2$, then if we write J for $u_1v_2-u_2v_1$, we shall have $n(uv_2-vu_2)=xJ$, $n(uv_1-vu_1)=-yJ$.

* In like manner, for a ternary quantic, the Jacobian of u, v, w is the locus of the double points of all curves of the system $u+kv+lw$ which have double points. And similarly for quantics with any number of variables.

Differentiating the first of these equations with regard to y, and the second with regard to x, we get

$$n(uv_{22} - vu_{22}) = xJ_2, \quad n(uv_{11} - vu_{11}) = -yJ_1.$$

It follows from the equations we have written, that any value α of x which makes both u and v vanish, will make not only J vanish but also its differentials J_1, J_2, and therefore $x - \alpha$ must be a square factor in J.

Or more directly thus: let $u = \beta\phi$, $v = \beta\psi$, where $\beta = lx + my$; then $u_1 = l\phi + \beta\phi_1$, $u_2 = m\phi + \beta\phi_2$, $v_1 = l\psi + \beta\psi_1$, $v_2 = m\psi + \beta\psi_2$; and $u_1v_2 - u_2v_1 = \beta^2(\phi_1\psi_2 - \phi_2\psi_1) + \beta l(\phi\psi_2 - \phi_2\psi) + \beta m(\phi_1\psi - \phi\psi_1)$, whence $(n-1)(u_1v_2 - u_2v_1)$

$$= (n-1)\beta^2(\phi_1\psi_2 - \phi_2\psi_1) + \beta(lx + my)(\phi_1\psi_2 - \phi_2\psi_1)$$
$$= n\beta^2(\phi_1\psi_2 - \phi_2\psi_1).$$

It follows from what has been said, that the discriminant of the Jacobian of u, v must contain R their resultant as a factor; since whenever R vanishes, the Jacobian has two equal roots. Thus in the case of two quadratics.

$$(a, b, c\chi x, y)^2, \quad (a', b', c'\chi x, y)^2,$$

the Jacobian is $(ab')x^2 + (ac')xy + (bc')y^2$,

whose discriminant is $4(ab')(bc') - (ac')^2$, which is the eliminant of the two quadratics. In the case of quantics of higher order, the discriminant of the Jacobian will, in addition to the resultant, contain another factor, the nature of which will appear from the following articles.

179. It has been said that we can always determine k, so that $u + kv$ shall have a square factor. But since two conditions must be fulfilled, in order that $u + kv$ may have a cube factor, k cannot be determined so that this shall be the case unless a certain relation connect the coefficients of u and v. *This condition will be of the order* $3(n-2)$ *in the coefficients both of* u *and* v.

If $(x - \alpha)^3$ be a factor in $u + kv + lw$, $x - \alpha$ will be a factor in the three second differential coefficients, or $x = \alpha$ will satisfy the equations

$$u_{11} + kv_{11} + lw_{11} = 0, \quad u_{12} + kv_{12} + lw_{12} = 0, \quad u_{22} + kv_{22} + lw_{22} = 0,$$

whence eliminating k and l, $x = \alpha$ will satisfy the equation

$$\begin{vmatrix} u_{11}, & v_{11}, & w_{11} \\ u_{12}, & v_{12}, & w_{12} \\ u_{22}, & v_{22}, & w_{22} \end{vmatrix} = 0.$$

If then we use the word treble-point in a sense analogous to that in which we used the word double-point (Art. 177), we see that the equation which has been just written gives the treble points of the system $u + kv + lw$; and since the equation is of the degree $3(n-2)$, there may be $3(n-2)$ such treble points. But we could find the number of treble points otherwise. Suppose we have formed the condition that $u + kv$ should admit of a treble point, and that this condition is of the order p in the coefficients of u. If in this condition we substitute for each coefficient (a) of u, $a + la''$, we get an equation of the degree p in l; and therefore p values of l will be found to satisfy it. In other words, p quantics of the system $u + kv + lw$ will have a treble point. It follows then from what has just been proved that $p = 3(n-2)$. And the same argument proves that the condition in question is of the order $3(n-2)$ in the coefficients of v.

This condition is evidently a combinant; for if it is possible to give such a value to k, as that $u + kv$ shall have a cube factor, it must be possible to determine k, so that $(u + mv) + kv$ shall have a cube factor.

180. If $u + kv$ have a cube factor $(x-\alpha)^3$, then the Jacobian of u and v will contain the square factor $(x-\alpha)^2$. For the two differentials $u_1 + kv_1$, $u_2 + kv_2$ will evidently contain this square factor, and therefore it will appear also in the Jacobian, which may be written $(u_1 + kv_1)\, v_2 - (u_2 + kv_2)\, v_1$. If then $S = 0$ be the condition that $u + kv$ may have a cube factor, S will be a factor in the discriminant of the Jacobian, since if $S = 0$ the Jacobian has two equal roots, and therefore its discriminant vanishes.

If R be the resultant, the discriminant of the Jacobian can only differ by a numerical factor from RS. For since the Jacobian is of the degree $2(n-1)$, its discriminant is of the degree $2\{2(n-1)-1\}$ in its coefficients, which are of the first order in the coefficients of both u and v. Now R is of the order

n in each set of coefficients, S of the order $3(n-2)$. Both these are factors in the discriminant; and it can have no other, since

$$n + 3(n-2) = 2\{2(n-1) - 1\}.$$

181. The discriminant of $u + kv$, considered as a function of k, will have a square factor whenever u and v have a common factor. In fact (Art. 111) the discriminant of $u + kv$ will be of the form $(a + ka')\phi + (b + kb')^2\psi$. But if u and v have a common factor, we can linearly transform u and v so that this factor shall be y, that is to say, so that both a and a' shall vanish. The discriminant will therefore have the square factor $(b + kb')^2$; and since the form of the discriminant is not affected by a linear transformation of the variables, it always has a square factor in the case supposed.

It follows that if we form the discriminant of $u + kv$, and then the discriminant of this again considered as a function of k, the latter will contain as a factor R the resultant of u and v. For it has been proved that when $R = 0$, the function of k has two equal roots, and therefore its discriminant vanishes. For example, the discriminant of a quadratic $ac - b^2$ becomes, by the substitution of $a + ka'$ for a, &c.,

$$(ac - b^2) + k(ac' + ca' - 2bb') + k^2(a'c' - b'^2),$$

whose discriminant is

$$4(ac - b^2)(a'c' - b'^2) - (ac' + ca' - 2bb')^2.$$

But this is a form in which, as was shewn by Boole, the resultant of the two quadratics $(a, b, c\chi x, y)^2$, $(a', b', c'\chi x, y)^2$ can be written (cf. Ex. 6, p. 24). This form, all the component parts of which are invariants, is sometimes more convenient than that given (Art. 178). In the case of quantics of higher order, the discriminant of the discriminant will have R as a factor, but will have other factors besides.

182. If u have either a cube factor or two distinct square factors, the discriminant of $u + kv$ will be divisible by k^2. For if the discriminant of u be Δ, that of $u + kv$ is

$$\Delta + k\left(a'\frac{d\Delta}{da} + b'\frac{d\Delta}{db} + \&c.\right) + \&c.$$

Now when u has a square factor Δ vanishes; and it appears from the expressions in Art. 114, that if either three roots of u are equal $\alpha = \beta = \gamma$, or two distinct pairs be equal $\alpha = \beta$, $\gamma = \delta$, then all the differentials of Δ, $\frac{d\Delta}{da}$, &c., vanish; and therefore the coefficient of k in the expression just given vanishes. The discriminant therefore contains k^2 as a factor. It is evident hence that if $u + av$ have a cube or two square factors, the discriminant of $u + kv$ will be divisible by $(k-a)^2$; since $u + kv$ may be written $u + av + (k-a)v$. If then, as before, $S = 0$ express the condition that the series $u + kv$ may include one quantic having a cube factor; and if $T = 0$ be the condition that it should include one having two square factors, both S and T will be factors in the discriminant with respect to k of the discriminant of $u + kv$. For we have just seen that the discriminant has a square factor if either $S = 0$ or $T = 0$. We proved in the last Article that the discriminant has R as a factor; and, in fact, the discriminant will be, as Prof. Cayley has observed, $RS^3 T^2$. I do not know whether there is any more rigid proof of this than that we see that there is no other case in which the discriminant of $u + kv$ has a square factor; that we find in the case of the third and fourth degrees that S and T enter in the form S^3, T^2; and that we can thus account for the order in general. For the discriminant of $u + kv$ is of the order $2(n-1)$ in k, and the coefficients are of the order $0, 1, \ldots, 2(n-1)$ in the coefficients of either quantic. The discriminant then with respect to k will be of the order $2(n-1)(2n-3)$ in the coefficients of either quantic. But R is of the order n, S of the order $3(n-2)$, and it will be proved in a subsequent lesson that T is of the order $2(n-2)(n-3)$, and

$$2(n-1)(2n-3) = n + 9(n-2) + 4(n-2)(n-3).$$

183. It was stated (Art. 176) that every combinant of u, v becomes multiplied by a power of $(\lambda\mu' - \lambda'\mu)$ when we substitute $\lambda u + \mu v$, $\lambda' u + \mu' v$ for u, v. It will be useful to prove otherwise that the eliminant of u, v has this property. First, let it be observed that if we have any number of quantics,

one of which is the product of several others, u, v, $ww'w''$, their resultant is the product of the resultants (uvw), (uvw'), (uvw''). For when we substitute the common roots of u, v in the last and multiply the results, we evidently get the product of the results of making the same substitution in w, w', w''. Again, the resultant of u, v, kw is the resultant of u, v, w multiplied by k^{mn}, since the coefficients of w enter into the resultant in the degree mn. If now $R(u, v)$ denote the resultant of u, v, which are supposed to be both of the same degree n, we have

$$\mu'^n R(\lambda u + \mu v, \lambda' u + \mu' v) = R(\lambda\mu' u + \mu\mu' v, \lambda' u + \mu' v)^*$$
$$= R\{(\lambda\mu' - \lambda'\mu) u, \lambda' u + \mu' v\} = (\lambda\mu' - \lambda'\mu)^n R(u, \lambda' u + \mu' v)$$
$$= (\lambda\mu' - \lambda'\mu)^n \mu'^n R(u, v),$$

whence $R(\lambda u + \mu v, \lambda' u + \mu' v) = (\lambda\mu' - \lambda'\mu)^n R(u, v)$.

By the same method it can be proved that the eliminant of $\lambda u + \mu v + \nu w$, $\lambda' u + \mu' v + \nu' w$, $\lambda'' u + \mu'' v + \nu'' w$ is $(\lambda\mu'\nu'')^n$ times that of u, v, w, and so on.

184. If U, V be functions of the orders m and n respectively in u, v, which are themselves functions of x, y of the order p, and if D be the result of eliminating u, v, between U, V; then the result of eliminating x, y between U, V will be D^p times the mn^{th} power of the resultant of u, v. For U may be resolved into the factors $u - \alpha v$, $u - \beta v$, &c., and V into $u - \alpha' v$, $u - \beta' v$, &c. And, Art. 183, the resultant of U, V will be the product of all the separate resultants $u - \alpha v$, $u - \alpha' v$. But one of these is $(\alpha - \alpha')^p R(u, v)$. There are mn such resultants. When therefore we multiply all together, we get the mn^{th} power of $R(u, v)$ multiplied by the p^{th} power of $(\alpha - \alpha')$ $(\alpha - \alpha'')$, &c. But this last is the eliminant of U, V with respect to u, v.

185. Similarly, let it be required to find the discriminant, with respect to x, y, of U, where U is a function of u, v. First, let it be remarked (see Art. 110) that the discriminant of the product of two binary quantics u, v is the product of the

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* The resultant of $u + kv$, v, being the same as the resultant of u, v, Art. 176, we next subtract μ times the second quantic from the first.

discriminants of u and v multiplied by the square of their resultant.

If then $U=(u-\alpha v)\,(u-\beta v)$ &c., the discriminant of U will be the product of the discriminants of $u-\alpha v$, $u-\beta v$, &c. by the square of the product of all the separate resultants $u-\alpha v$, $u-\beta v$. But, as before, any of these will be $(\alpha-\beta)^p\,R\,(u, v)$. If then m be the degree of U considered as a function of u, v; there will be $\frac{1}{2}m\,(m-1)$ separate resultants, and the square of the product of all will be $(\alpha-\beta)^{2p}\,(\alpha-\gamma)^{2p}$, &c. $\times\, R^{m(m-1)}\,(u, v)$. But $(\alpha-\beta)^2\,(\alpha-\gamma)^2$, &c. is the discriminant of U considered as a function of u, v. If then we call this Δ, we have proved that the product of the squares of the separate resultants is $\Delta^p R^{m(m-1)}$. Let us now consider the product of the discriminants of $u-\alpha v$, $u-\beta v$, &c.; this is the result of eliminating θ between the discriminant of $u-\theta v$, which is a quantic of the order $2\,(p-1)$ in θ and the quantic of the m^{th} order got by substituting $u=\theta v$ in U. Or this product has been otherwise represented by Dr. Sylvester. If a_0, b_0 be the coefficients of x^p in u, v, then (Art. 108) the resultant of $u-\alpha v$, $u_1-\alpha v_1$ will be $a_0-\alpha b_0$ times the discriminant of $u-\alpha v$. But

$$R\,(u, v)\,R\,(u-\alpha v, u_1-\alpha v_1)=R\,(u-\alpha v, u_1 v-\alpha v v_1)=R\,(u-\alpha v, u_1 v-u v_1).$$

Now (Art. 178) $p\,(u_1 v-u v_1)=yJ$ where J is $u_1 v_2-u_2 v_1$, and $R\,(u-\alpha v, y)$ is $a_0-\alpha b_0$. It appears thus that the discriminant of $u-\alpha v$ differs only by a numerical factor from the resultant of $(u-\alpha v, J)$ divided by $R\,(u, v)$. The product then of all the discriminants will be the—resultant of J and the product $u-\alpha v$, $u-\beta v$, &c., in other words, the resultant of U, J—divided by the m^{th} power of $R\,(u, v)$. Thus we have Dr. Sylvester's result (*Comptes Rendus*, LVIII., 1078) that the discriminant of U with respect to $x:y$ is $\Delta^p R\,(u, v)^{m(m-2)}\,R\,(U, J)$. But it will be observed that the result expressed thus is not in its most reduced form since $R\,(U, J)$ contains the factor $R\,(u, v)^m$.

186. We have next to see what corresponds in the case of ternary and quaternary quantics to the theory just explained for systems of binary quantics. Let then u and v be two ternary quantics, and let us suppose that we have formed the

discriminant of $u + kv$. Then (for certain relations between the quantics u, v) this discriminant considered as a function of k will have a square factor. In the first place the discriminant will have a square factor, if the curves represented by u and v touch each other. For we have seen (Art. 117) that if the equation of a curve be $az^n + bz^{n-1}x + cz^{n-1}y + \&c. = 0$, its discriminant is of the form $a\theta + b^2\phi + bc\psi + c^2\chi$. The discriminant then of $u + kv$ will be of the form $(a + ka')\,\theta + (b + kb')^2\,\phi + \&c.$ But if we take for the point xy, a point common to u and v, both a and a' will vanish; and if we take the line y for the common tangent, both b and b' vanish; and the discriminant will be of the form $(c + kc')^2\,\chi$; and therefore will always have a square factor in the case supposed.

187. Again, the discriminant will have a square factor if u have either a cusp or two double points. The vanishing of the discriminant Δ of a ternary quantic gives the condition that u_1, u_2, u_3 shall have a common system of values. If, however, u have either two double points, or a cusp, u_1, u_2, u_3 will have two systems of common values, distinct or coincident, and therefore (Art. 103) not only will Δ vanish, but also its differentials with respect to all the coefficients of u. The discriminant then of $u + kv$ being in general $\Delta + k\left(\frac{a'd\Delta}{da} + \frac{b'd\Delta}{db} + \&c.\right) + \&c.$ will in this case be divisible by k^2. And as in Art. 182, it will be divisible by $(k - \alpha)^2$ if the curve $u + \alpha v$ have either a cusp or two double points.

Let then $R = 0$ be the tact-invariant of u and v, that is to say, the condition that the two curves should touch; $S = 0$ the condition that in the system of curves $u + kv$ shall be included one having a cusp; and $T = 0$ the condition that there shall be included one having two double points. It has been proved that R, S, T are all factors in the discriminant of the discriminant of $u + kv$, considered as a function of k. In fact this discriminant will be RS^3T^2. For an investigation of the orders of R, S, T, when both curves are of the same degree, see *Higher Plane Curves*, Art. 399.

The tact-invariant R is of the order $3n(n-1)$ in the coeffi-

cients, S of the order $12(n-1)(n-2)$, and T of the order $\frac{3}{2}(n-1)(n-2)(3n^2-3n-11)$: the discriminant of $u+kv$ in regard to $x:y$ is of the order $3(n-1)^2$ and the discriminant of this with regard to k of the order $3(n-1)^2(3n^2-6n+2)$, and we have identically

$$3(n-1)^2(3n^2-6n+2)$$
$$=3n(n-1)+36(n-1)(n-2)+3(n-1)(n-2)(3n^2-3n-11),$$

showing that the order of the discriminant is equal to that of RS^3T^2.

188. The theorem given, Art. 110, for the discriminant of the product of two binary quantics cannot be extended to ternary quantics; for the discriminant of the product of two will, in this case, vanish identically. In fact, the discriminant is the condition that a curve shall have a double point; and a curve made up of two others has double points; namely, the intersections of the component curves. Or, without any geometrical considerations, the discriminant of uv is the condition that values of the variables can be found to satisfy simultaneously the differentials uv_1+vu_1, uv_2+vu_2, &c. But these will all be satisfied by any values which satisfy simultaneously u and v; and such values can always be found when there are more than two variables.

But the theorem of Art. 110 may directly be extended to tact-invariants. The condition that u shall touch a compound curve vw will evidently be fulfilled if u touch either v or w, or go through an intersection of them. For an intersection counts, as has been said, as a double point on the complex curve; and a line going through a double point of a curve is to be considered doubly as a tangent. Hence if $T(u, v)$ denote the tact-invariant of u, v, we have

$$T(u, vw) = T(u, v)\, T(u, w)\, \{R(u, v, w)\}^2,$$

when $R(u, v, w)$ is the resultant of u, v, w. And the result may be verified by comparing the order in which the coefficients of u, v, or w occur in these invariants. Thus, for the coefficients of u, we have

$$(n+p)(n+p+2m-3) = n(n+2m-3)+p(p+2m-3)+2np.$$

189. The theory of the tact-invariants of quaternary quantics is given in *Geometry of Three Dimensions*, p. 544; and there is not the least difficulty in forming the general theory of the class of invariants we have been considering, to which Dr. Sylvester proposes to give the name of *Osculants*. Let there be i quantics, U, V, W, &c. in k variables; then the osculant is the condition that for the same system of values which satisfy U, V, &c. the tangential quantics $xU_1' + yU_2' + \&c.$, &c. shall be connected by an identical relation

$$\lambda(xU_1' + \&c.) + \mu(xV_1' + \&c.) + \nu(xW_1' + \&c.) + \&c. = 0.$$

In other words, the osculant is the condition that the equations $U=0$, $V=0$, &c., and also the system

$$\left\| \begin{matrix} U_1, & U_2, & U_3, & \&c. \\ V_1, & V_2, & V_3, & \&c. \\ W_1, & W_2, & W_3, & \&c. \end{matrix} \right\| = 0$$

can be simultaneously satisfied. This latter system having k columns and i rows is equivalent to $k-i+1$ equations; therefore this system combined with the given i equations is apparently equivalent to $k+1$ equations in k variables. It is really, however, only equivalent to k equations; for writing $U=0$ in the form $xU_1 + yU_2 + \&c. = 0$, and similarly for V, &c, we see that when the system of determinants is satisfied, and all but one of the equations $U=0$, $V=0$, $W=0$, &c., the remaining one must be satisfied also. The system then being equivalent to k equations in k variables cannot be simultaneously satisfied unless a certain condition be fulfilled. The order of this condition, in the coefficients of U, is found by the same method as in *Geometry of Three Dimensions*. We write for U, $U+\lambda u$, and we examine how many values of the variables can simultaneously satisfy the $i+1$ equations V, W, &c., and the system equivalent to $k-1$ equations

$$\left\| \begin{matrix} U_1, & U_2, & U_3, & \&c. \\ u_1, & u_2, & u_3, & \&c. \\ V_1, & V_2, & V_3, & \&c. \\ W_1, & W_2, & W_3, & \&c. \end{matrix} \right\| = 0.$$

The order of the $i-1$ equations V, W, &c, is the product of their degrees n, p, &c.; and the order of the osculant in the coefficients of U is the product of this number by the order of the system of determinants, which is found by the rule given in a subsequent Lesson on the order of systems of equations.

When we are given but one quantic, the osculant is the discriminant; when we are given k quantics in k variables, the osculant is the resultant. The theorem of Art. 110 may be extended to osculants in general; viz. that if we form the osculant of $k-1$ quantics in k variables, and if the last be the product of two quantics U, V, then the osculant of the entire system will be the product of the osculant of the system of the other $k-2$ with U, that of the system of $k-2$ with V, and the square of the resultant of all the quantics.

190. We have already seen (Art. 151) how the invariants and covariants of a single quantic are derived from those of a system of quantics in the same number of variables; and we wish now to point out how the invariants and covariants of a single quantic are connected with those of a system of quantics in a greater number of variables. Suppose, in fact, we had two ternary quantics, geometrically denoting two curves, we can, by eliminating one variable, obtain a binary quantic satisfied by the points of intersection of these curves; and it is evident, geometrically, that the invariants of the binary quantic (expressing the condition, for instance, that two of these points should coincide, or should have to each other some permanent relation) must also be invariants of the system of two ternary quantics. Conversely, we may consider any binary quantic as derived from a system of two ternary quantics; for we have only to assume $X=\phi(x, y)$, $Y=\psi(x, y)$, $Z=\chi(x, y)$, equations which in themselves imply, by elimination of x and y, one fixed relation between X, Y, Z, and from which, combined with the given binary quantic equated to zero, we can obtain a second such relation. The simplest example of such a transformation is that investigated by Mr. Burnside *Quarterly Journal* x. (1870) p. 211; (compare *Conics*, note pp. 386–7), where ϕ, ψ, χ

are quadratic* functions of x and y. The substitution is then reducible by linear transformation to $X=x^2$, $Y=2xy$, $Z=y^2$, giving the fixed relation $4XZ-Y^2=0$. By making these substitutions for x^2 &c. in a binary quantic of even degree, we have at once a second relation between X, Y, Z; if the quantic be of odd degree, it can be brought to an even degree by squaring. The resulting relation is obviously not unique, but is of the form $\phi_{2m}+\phi_{2m-2}(4XZ-Y^2)$, where ϕ_{2m} is any one form of the relation, and the coefficients in ϕ_{2m-2} are arbitrary. Geometrically, the binary quantic of the m^{th} degree is thus made to represent m points on a conic, determined when m is even by the intersection of the conic with a curve of the order $\frac{1}{2}m$, and when m is odd with a curve of order m touching the conic in m points. Among these forms there is always one whose invariants and covariants are also invariants and covariants of the given binary quantic.†

Thus the binary quadratic $ax^2+2bxy+cy^2$ is replaced by the system $aX+bY+cZ$, $4XZ-Y^2$, and geometrically denotes the two points of intersection of a line with a conic. The discriminant of the quadratic is also an invariant of the system; that whose vanishing expresses the condition that the line shall touch the conic. So, in like manner, the system of two binary quadratics $ax^2+2bxy+cy^2$, $a'x^2+2b'xy+c'y^2$, gives rise to the system of a conic and two lines. The invariant of the binary system $ac'+ca'-2bb'$ (Art. 151) is also an invariant of the ternary system; viz., its vanishing expresses that the lines are harmonic polars with respect to the conic.

If three lines L, M, N be mutually harmonic polars with respect to a conic, we know (*Conics*, Art. 271) that the equation of the conic may be written in the form $V\equiv lL^2+mM^2+nN^2=0$, whence we infer immediately that if three binary quadratics be

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* If linear functions had been taken, the transformation could be reduced to $X=x$, $Y=y$, $Z=0$, and the binary quantic of the n^{th} degree would represent n points on the line Z (see Art. 177).

† This form can be found by operating on $\phi_{2m}+\phi_{2m-2}(4XZ-Y^2)$ with the form reciprocal to $4XZ-Y^2$, viz. $\frac{d^2}{dXdZ}-\frac{d^2}{dY^2}$, and equating to zero the coefficients of every term in the result.

connected in pairs by the relation $ac' + ca' - 2bb' = 0$, their squares are connected by an identical relation $lL^2 + mM^2 + nN^2 = 0$, for V vanishes identically when we return to binary quantics.

To the Jacobian of two binary quadratics answers, for the system of two lines and a conic, the line $2(ab')X + (ac')Y + 2(bc')Z$, which is also a covariant of that system. In fact, it is the polar with respect to the conic of the intersection of the two lines.

More generally, the Jacobian of any system u, v will be transformed into the Jacobian of the system formed by U, V, and the fixed conic. For let u_1, u_2, u_3, denote the differentials of u with respect to X, Y, Z, which, it will be remembered, denote x^2, $2xy$, y^2 respectively, then the Jacobian is

$$\begin{vmatrix} xu_1 + yu_2, & xu_2 + yu_3 \\ xv_1 + yv_2, & xv_2 + yv_3 \end{vmatrix} = \begin{vmatrix} y^2, & -xy, & x^2 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix} = \begin{vmatrix} Z, & -\tfrac{1}{2}Y, & X \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix},$$

but the terms in the first line are proportional to the differentials of $4XZ - Y^2$.

The same method being applied to the discussion of the biquadratic, it is found to be equivalent to the system of two conics, viz. the fixed conic $4XZ - Y^2$, and the conic

$$aX^2 + cY^2 + eZ^2 + 2dYZ + 2cZX + 2bXY = 0,$$

the discriminant of the latter conic being also an invariant of the quartic (Art. 171). So again the system of two binary quartics is equivalent to a system of three conics. We shall have occasion in the next Lesson to give further illustrations of this method: it has been applied by Mr. W. R. W. Roberts to the system of two cubics which involve properties of a twisted cubic, *Proc. Lond. Math. Soc.* vol. XIII. (1881), and the relation between binary and quaternary forms is developed with the general symbolic formulæ in a paper by Dr. Lindemann, *Math. Ann.* (1884) XXIII. p. 111, &c.

LESSON XVII.

APPLICATIONS TO BINARY QUANTICS.

191. HAVING now explained the most essential parts of the general theory, we wish to illustrate its application by enumerating the different invariants and covariants of binary quantics for the lower degrees. If S and T be invariants of the same degree, or covariants of the same degree and order, and k any numerical factor, then $S+kT$, which will of course be also an invariant or covariant, will not be reckoned in our enumeration as distinct from the invariants S and T. And, generally, any invariant or covariant which can be expressed as a rational and integer function of other invariants and covariants of the same or lower degrees is said to be reducible, and will not be considered as distinct from these latter functions. It is otherwise if the expression be not rational and integer. Thus, if S be an invariant of the second and T of the third degree, then though S^3+kT^2 would not be regarded as a new invariant, yet if it be a perfect square, and we have $R^2=S^3+kT^2$, we count R as a new invariant distinct from S and T, and call it irreducible.

It was proved in Art. 121 that a binary quantic has $n-3$ *absolute* invariants, and in Art. 122 that from any two ordinary invariants an absolute invariant can be deduced. We should infer, therefore, that the number of independent ordinary invariants is one more than the number of absolute invariants; or, in other words, that a binary quantic of the n^{th} order has $n-2$ invariants, in terms of which every other invariant can be expressed. But as it does not follow that the expression is necessarily rational, we do not in this way obtain any limit to the number of irreducible invariants. And so as regards the covariants (including in this expression the invariants) we shall presently see that for a quantic of the n^{th} order there are, inclusive of the quantic itself, n covariants, such that

every other covariant multiplied by a power of the quantic is equal to a rational and integer function of the n covariants; thus, each such other covariant is a rational, but not an integer function of the n covariants; and we do not hereby obtain any limit to the number of the irreducible covariants. We have stated (Art. 145) the method by which Prof. Cayley originally attempted to determine the number of distinct covariants and invariants. He did not at the time succeed in obtaining any limit to their number for quantics above the fourth order. Subsequently Gordan proved (see *Crelle*, vol. LXIX., or Clebsch, *Theorie der binären algebraischen Formen*, p. 255, also his Programm for the University of Erlangen, *Ueber das Formensystem binärer Formen*, Leipzic, 1875), that for a binary quantic or system of binary quantics, the number of distinct invariants and covariants is always finite; and he has given a process by which when we have the complete system of invariants and covariants for a quantic of any degree, we can find the system corresponding to the next higher degree. His proof, which is founded on an analysis of the different possible expressions by the symbolical method explained Lesson XIV., will be found in a subsequent Lesson on that method. Later still, Prof. Sylvester has investigated the whole subject by Prof. Cayley's method, founded on the theory of the partition of numbers, in various memoirs in the *Comptes Rendus*, vol. LXXXIV. pp. 974–5, 1113–6, &c., and subsequent volumes, as also in the *American Journal of Mathematics*.

192. It will be convenient to bear in memory what was proved, Art. 147, that a covariant is completely known when its leading coefficient, or, as we have there called it, its source, is known; this coefficient being any function of the differences of the roots of the quantic.* Thus take the quantic $(a, b, c \ldots)(x, y)^n$, we know that, in the case of the quadratic, $(ac - b^2)$ is an invariant; and if we desire to form the covariant

* Such functions have been called semi-invariants or seminvariants, as they remain unaltered (see Art. 62) when we substitute $x + \lambda$ for x, but not necessarily when we substitute $y + \lambda$ for y; and as they satisfy one of the differential equations given in that article, but not necessarily the other.

$(ac-b^2)\,x^p+$&c., having this leading term, we observe that the weight of the given source and its degree in the coefficients are each $=2$, so that writing $\theta=2$ in the formula (Art. 147) $2=\frac{1}{2}(n\theta-p)$ we have $p=2(n-2)$. The other coefficients are found by the method explained in that article; thus the covariant is found to be

$$(ac-b^2)\,x^{2(n-2)}+(n-2)\,(ad-bc)\,x^{2n-5}y$$
$$+(n-2)\{(bd-c^2)+\tfrac{1}{2}(n-3)\,(ae-c^2)\}\,x^{2n-6}y^2+\&c.$$

It follows also from what has been stated in the article referred to, that any algebraic relation between the sources of different covariants implies a corresponding relation between the covariants themselves.

Prof. Cayley has used this principle in attempting to form the complete system of the covariants of a binary quantic; and though it does not lead to any general theory it furnishes the most elementary and satisfactory proof of the numbers of concomitants for functions of the first four degrees. The leading coefficient of any covariant being a function of the differences must (Art. 62) satisfy the differential equation

$$(ad_b+2bd_c+3cd_d+\&c)\,U=0:$$

and we assume that U is a rational and integer function of a, b, c, &c. Now, if we solve the partial differential equation, we find that U must be a function of

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ a^3e-4a^2bd+6ab^2c-3b^4,\ \&c.,$$

where the law of formation of the successive terms is obvious; and, in fact, the covariants of which these terms are the leaders are each the Jacobian of the preceding covariant in the series, combined with the original quantic. We shall refer to these quantities as L_1, L_2, L_3, &c. and we see that the leading coefficient of any covariant must be a function of these quantities: and it must of course be a rational function of them. The question is whether there are any rational, but not integer functions of L_1, L_2, L_3, &c., which are rational and integer functions of a, b, c; and a little consideration shows that the only admissible form is that of a rational and integer function divided by a power of L_1, that is a. For, the leading

coefficient in question is a rational function of the coefficients $(a, b, c, \ldots)$; and if we make in it $b = 0$, it becomes a rational function of a, c, d, &c., and by multiplying by a suitable power of a it can be made an integer function of a, ac, a^2d, a^3e, &c. But these are the values of L_2, L_3, &c. on the supposition of $b = 0$. Thus we see that the leader of any covariant can only be the quotient by a power of a of an integer function of these n quantities. Conversely, the problem of finding all possible covariants is the same as that of finding the new functions which arise when rational and integer functions of L_2, L_3, &c. are formed which are divisible by a. To find these functions we make $a = 0$ in L_2, L_3, &c. and eliminate b between any pair; we thus get a function of L_2, &c. which vanishes on the supposition of $a = 0$, and therefore is divisible by a power of a. By performing the division we obtain the leader of a new covariant. This again may be treated in like manner, by putting $a = 0$ and examining whether it be possible to eliminate the remaining coefficients. This method will be better understood from the applications which will be made presently.

It is obvious that the same considerations apply to the still simpler forms—of lowest degrees—of particular integrals of the partial differential equation a, $ac - b^2$, $a^2d - 3abc + 2b^3$, $ae - 4bd + 3c^2$, $a^2f - 5abe + 2acd - 6bc^2 + 8b^2d$, &c. of which the second, fourth, &c. are the successive quadrinvariants of even quantics as they arise, and the third, fifth, &c. are the sources of the evectants of the successive quartinvariants of the corresponding odd quantics as they arise. See Art. 142.

193. We have already stated the principal points in the theory of the quadric form $(a, b, c\chi x, y)^2$. Since there are but two roots and only one difference, there can be no function of the differences of the roots but a power of this difference; and the odd powers, not being symmetrical functions of the roots of the given quadratic, cannot be expressed rationally in terms of its coefficients. It thus immediately follows that the quadric has no covariants other than the quantic itself, and no invariant other than the powers of the discriminant, $ac - b^2$,

which is proportional to $(\alpha - \beta)^2$. We have already shewed (Art. 157) that it follows, by Hermite's law of reciprocity, that only quantics of even degree can have invariants of the second order in the coefficients. These are the system whose symbolical form is $\overline{12}^n$, explained Art. 153,

$$ac - b^2,\ ae - 4bd + 3c^2,\ ag - 6bf + 15ce - 10d^2,\ \&c.$$

If we make $y = 1$ in the quadratic it denotes geometrically a system of two points on the axis of x, and the vanishing of the discriminant expresses the condition that these points should coincide, Art. 177.

System of two quadrics. This system

$$(a, b, c\mathcal{X}x, y)^2,\ (a', b', c'\mathcal{X}x, y)^2,$$

has the invariant $\overline{12}^2$ or $ac' + ca' - 2bb'$. When each quantic is taken to represent a pair of points in the manner just stated, the vanishing of this invariant expresses the condition (see *Conics*, Art. 332) that the four points shall form a harmonic system, the two points represented by each quantic being conjugate to each other. We have also proved (Art. 177) that the covariant $\overline{12}$ (or the Jacobian of the system) represents the foci of the system in involution determined by the four points.

It is easy to see, as in Art. 169, or by *Conics*, Art. 333, that the Jacobian may be written in the form

$$J(u, v) = \begin{vmatrix} y^2, & -xy, & x^2 \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}.$$

Now by the ordinary rule for multiplication of determinants we have

$$\begin{vmatrix} y^2, & -xy, & x^2 \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} \times \begin{vmatrix} x^2, & 2xy, & y^2 \\ c, & -2b, & a \\ c', & -2b', & a' \end{vmatrix} = \begin{vmatrix} 0, & u, & v \\ u, & 2D, & \Delta \\ v, & \Delta, & 2D' \end{vmatrix},$$

or

$$2J^2 = -2u^2D' + 2uv\Delta - 2v^2D,$$

where J denotes the Jacobian, D and D' the discriminants of the quantics, and Δ the intermediate invariant $ac' + ca' - 2bb'$.

This equation includes the theorem stated Art. 190, for the case $\Delta = 0$. The equation just given may also be easily verified by means of the canonical form. We have seen (Art. 177) that there are two values of k, for which $u + kv$ is a perfect square, and if these squares be x^2 and y^2 the system may be written $ax^2 + cy^2$, $a'x^2 + c'y^2$, or more simply $x^2 + y^2$, $ax^2 + cy^2$. We have then $D = 1$, $D' = ac$, $\Delta = a + c$, $J = (c - a)\,xy$, by means of which values the preceding equation is at once verified. So again the Jacobian of u, J is for the canonical form $\frac{1}{2}(c - a)(x^2 - y^2)$, and therefore is in general $\frac{1}{2}\Delta u - Dv$. The invariant Δ taken between u and J vanishes identically, as is geometrically evident.

All other invariants or covariants of a system of two quadrics may be expressed in terms of u, v, J, D, D', Δ.

Thus the eliminant

$$(ac' - ca')^2 + 4\,(ba' - ab')\,(bc' - cb'),$$

may also be written in the form

$$(ac' + ca' - 2bb')^2 - 4\,(ac - b^2)\,(a'c' - b'^2).$$

In other words, the eliminant is the discriminant either of the Jacobian

$$(ab')\,x^2 + (ac')\,xy + (bc')\,y^2,$$

or of $\quad (ac - b^2)\,\lambda^2 + (ac' + ca' - 2bb')\,\lambda\mu + (a'c' - b'^2)\,\mu^2.$

The former expression is linearly transformed into the latter by the substitution $\lambda b + \mu b'$, $-(\lambda a + \mu a')$ for x and y.

System of three quadrics

$$(a, b, c\,\text{X}\,x, y)^2,\ (a', b', c'\,\text{X}\,x, y)^2,\ (a'', b'', c''\,\text{X}\,x, y)^2.$$

This system has, in addition to the invariants and covariants corresponding to the respective pairs of quadrics, the determinant $\overline{23}.\overline{31}.\overline{12}$,

$$R = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

whose vanishing expresses the condition that the three pairs of points represented by the quadratics shall form a system in involution (Ex. 7, p. 25, also *Conics*, p. 310). This invariant

formed for u, v, J is another expression for the eliminant of u and v.

The expression found for J^2 of two quadrics may be generalized, if in the second determinant we write in the second and third rows c'', $-2b''$, a''; c''', $-2b'''$, a'''. We thus find that if there be four binary quadrics u_1, u_2, u_3, u_4,

$$2J_{12}J_{34} = \begin{vmatrix} 0, & u_1, & u_2 \\ u_3, & D_{13}, & D_{23} \\ u_4, & D_{14}, & D_{24} \end{vmatrix}.$$

We get, similarly, an expression for the product of two invariants $R_{123}R_{456}$,

$$2R_{123}R_{456} = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} \begin{vmatrix} c_4, & -2b_4, & a_4 \\ c_5, & -2b_5, & a_5 \\ c_6, & -2b_6, & a_6 \end{vmatrix} = \begin{vmatrix} D_{14}, & D_{15}, & D_{16} \\ D_{24}, & D_{25}, & D_{26} \\ D_{34}, & D_{35}, & D_{36} \end{vmatrix},$$

To these formulæ may be added the following, the truth of which is easily seen,

$$u_1J_{23} + u_2J_{31} + u_3J_{12} = 0,$$

$$u_1R_{234} - u_2R_{341} + u_3R_{412} - u_4R_{123} = 0,$$

$$R_{123}u_1 = D_{11}J_{23} + D_{12}J_{31} + D_{13}J_{12}, \text{ \&c.}$$

$$2R_{123}J_{23} = (D_{22}D_{33} - D_{23}^2)\,u_1 + (D_{32}D_{13} - D_{12}D_{33})\,u_2 + (D_{12}D_{23} - D_{13}D_{22})\,u_3, \text{ \&c.}$$

From the linear relations connecting the u's and J's follow the quadric identities

$$0 = D_{11}J_{23}^2 + D_{22}J_{31}^2 + D_{33}J_{12}^2 + 2D_{23}J_{31}J_{12} + 2D_{31}J_{12}J_{23} + 2D_{12}J_{23}J_{31},$$

$$0 = \begin{vmatrix} u_1, & u_2, & u_3, & 0 \\ D_{11}, & D_{12}, & D_{13}, & u_1 \\ D_{21}, & D_{22}, & D_{23}, & u_2 \\ D_{31}, & D_{32}, & D_{33}, & u_3 \end{vmatrix}.$$

Thus a system of three binary quadrics at once gives rise to a conic and their three Jacobians to its reciprocal form. The equation of the conic is referred to any line and two others through its pole when the three binary quadrics are any two arbitrarily taken and their Jacobian. The conic breaks up into right lines if the three binary quadrics form an involution.

194. It is to be remarked that, by means of Euler's theorem for homogeneous functions, the theory of those covariants of any quantic, the expression of which contains differential coefficients in not higher than the second degree, reduces itself to the theory of the quadric; and so every relation between the covariants of a quadric has answering to it a relation between such covariants of a quantic in general. Similarly, a relation between covariants of a cubic gives a relation between general covariants not involving differential coefficients in more than the third degree, and so on. Thus, the expression obtained for the square of the Jacobian of two quadrics gives the identical relation

$$\{(ax+by)(b'x+c'y)-(a'x+b'y)(bx+cy)\}^2$$
$$=-(ac-b^2)(a'x^2+2b'xy+c'y^2)^2+\&c.$$

But if a, b, c; a', b', c' denote the second differential coefficients of any two quantics, we have

$$ax+by=(n-1)u_1,\ a'x^2+2b'xy+c'y^2=n'(n'-1)v,\ \&c.,$$

whence we have an expression for the square of the Jacobian of any two quantics

$$(n-1)^2(n'-1)^2J^2=-n'^2(n'-1)^2Hv^2$$
$$+nn'(n-1)(n'-1)\Delta uv-n^2(n-1)^2H'u^2,$$

where H denotes the Hessian $ac-b^2$ and Δ, as before, the covariant $ac'+ca'-2bb'$.

So again, since the Jacobian involves only differential coefficients in the first degree, the Jacobian of J, u, involves them only in the second, and therefore can be expressed by means of the theory of the quadric. Writing L, M for the first differential coefficients, we have

$$J=LM'-L'M;$$

$$J(J,u)=\{aMM'-b(LM'+L'M)+cLL'\}-\{a'M^2-2b'LM+c'L^2\}.$$

But the values of the two members of the right-hand side of the equation are immediately found by the canonical form of the quadric, and are respectively

$$\frac{n'}{n-1}Hv,\ \text{and}\ \frac{n}{n-1}\Delta u-\frac{n'(n'-1)}{(n-1)^2}Hv,$$

whence $$J(J, u) = \frac{n'(n+n'-2)}{(n-1)^2} Hv - \frac{n}{n-1} \Delta u.$$

195. *The cubic.* We come next to the concomitants of the cubic

$$U = (a, b, c, d)(x, y)^3.$$

It has but one invariant (Art. 167), viz. the discriminant

$$D = a^2d^2 + 4ac^3 - 6abcd + 4db^3 - 3b^2c^2.$$

If the cubic were written without binomial coefficients, the discriminant would be $27a^2d^2 + 4ac^3 - 18abcd + 4db^3 - b^2c^2$. It is to be noted that the function here written is, with sign changed, the product by a^4 of the squares of the differences of the roots of the cubic. A useful expression may be derived from the last remark. Consider the three quantities $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, they are the roots of a cubic for which $a=1$, $b=0$, $c=-\frac{1}{2}\{(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2\}$, $d=(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$. Hence

$$(2\alpha-\beta-\gamma)^2(2\beta-\gamma-\alpha)^2(2\gamma-\alpha-\beta)^2$$
$$=\tfrac{1}{2}\{(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2\}^3 - 27(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2.$$

The Hessian $\overline{12^2}$ or H, is

$$(ac-b^2)x^2 + (ad-bc)xy + (bd-c^2)y^2 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ y^2, & -xy, & x^2 \end{vmatrix}.$$

This has the same discriminant as the cubic itself (Art. 167).

The cubic covariant $\overline{12^2}.\overline{13}$, or the evectant of the discriminant, which we call J, is (see Art. 142)

$$(a^2d-3abc+2b^3, abd+b^2c-2ac^2, 2b^2d-acd-bc^2, 3bcd-ad^2-2c^3)(x, y)^3,$$

which may also be written in the determinant form

$$\begin{vmatrix} x^3, & 3x^2y, & 3xy^2, & y^3 \\ c, & -2b, & a, & 0 \\ d, & -c, & -b, & a \\ 0, & d, & -2c, & b \end{vmatrix}.$$

This cubic may be geometrically represented as follows:—If we take the three points represented by the cubic itself, and take the fourth harmonic of each with respect to the other two, we

get three new points which will be the geometrical representation of the covariant in question. This theorem is suggested by its being evident on inspection, that if the given cubic take the form $xy(x+y)$, then $x-y$ will be a factor in the covariant, as appears by making $a=d=0$, $b=c=1$ in its equation. But $x+y$, $x-y$ are harmonic conjugates with respect to x and y. Now, if α, β, γ, δ denote the distances from the origin of four points on the axis of x, any harmonic or anharmonic relation* between them is expressed by the ratio of the products $(\alpha-\beta)(\gamma-\delta)$ and $(\alpha-\gamma)(\beta-\delta)$: and this ratio (see Art 136) is unaltered by a linear transformation; that is, when for each distance α we substitute $\frac{\lambda\alpha+\mu}{\lambda'\alpha+\mu'}$. Such relations, then, being unaltered by linear transformation, if proved to exist in one case, exist in general. We find that the other factors in the evectant of $xy(x+y)$ are $x+2y$, $2x+y$, so that our result may be written symmetrically, that the evectant of xyz (where x, y, z are connected by the linear relation $x+y+z=0$) is $(y-z)(z-x)(x-y)$. These considerations lead us to the expression for the factors of the covariant in terms of the roots of the given cubic: for if δ be the distance from the origin of the point conjugate to α with respect to β and γ; solving for δ from the equation $\frac{2}{\alpha-\delta}=\frac{1}{\alpha-\beta}+\frac{1}{\alpha-\gamma}$ we get $\delta=\frac{\alpha\beta+\alpha\gamma-2\beta\gamma}{2\alpha-\beta-\gamma}$, whence the covariant must be

$$a^3\{(2\alpha-\beta-\gamma)x+(2\beta\gamma-\alpha\beta-\alpha\gamma)y\}\{(2\beta-\alpha-\gamma)x$$

$$+(2\gamma\alpha-\beta\gamma-\beta\alpha)y\}\{(2\gamma-\alpha-\beta)x+(2\alpha\beta-\gamma\alpha-\gamma\beta)y\}=-27J,$$

as may be verified by actual multiplication and substitution in terms of the coefficients of the equation.

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* The anharmonic ratio of four quantities has any of the six values, according to the order assumed, λ, $\frac{1}{\lambda}$, $1-\lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda-1}{\lambda}$, $\frac{\lambda}{\lambda-1}$, which are in general all different. They may come to have equal values either if $\lambda=1$ when two values of the quantities are equal and the other values of the anharmonic ratio are 0 and ∞; or if $\lambda=-1$, when the quantities form a *harmonic* series, and the other values of the anharmonic ratio are 2 and $\frac{1}{2}$; or if $\lambda^2-\lambda+1=0$, when the quantities form an *equi-anharmonic* series, three values of the anharmonic ratio are one imaginary cube root of -1, and three its other imaginary cube root.

Similarly for the quadric covariant, if ω be an imaginary cube root of unity and we solve for δ from

$$(\alpha-\gamma)(\delta-\beta)+\omega(\beta-\gamma)(\delta-\alpha)=0,$$

we get $$(\alpha+\omega\beta+\omega^2\gamma)\delta+\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta=0$$

to determine a distance equi-anharmonic to α, β, γ. Hence, we have the covariant

$$\{(\alpha+\omega\beta+\omega^2\gamma)x+(\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta)y\}$$
$$\times\{(\alpha+\omega^2\beta+\omega\gamma)x+(\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta)y\},$$

double this is found to be $=\Sigma(x-\alpha)^2(\beta-\gamma)^2$, which, multiplied by a^2 and expressed in terms of the coefficients, $=-18H$.

196. We can now see that our list of covariants is complete. The leading coefficient of any covariant is a function of the differences $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$. Since the sum of these quantities is zero, any symmetric function of them can be expressed in terms of the sum of their squares, and their continued product. But since this product is only half symmetrical with respect to the roots of the given cubic, that is to say, is liable to change sign by an interchange of the roots, it can enter only by its square into a function expressible in terms of the coefficients. We thus see, that if the leader be a symmetric function of the differences, the covariant can be expressed as a rational function of U, H, D. But there is another function, viz. the product of the differences $(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$, which though only half symmetrical with respect to the differences, is symmetrical with respect to the roots of the given quantic. This is the leading term of the covariant J. But obviously the square of this function can be expressed in terms of the sum of squares, and product of differences. The expression has, in fact, been given in the last article. It is easy to prove, that in the case of the cubic written with binomial coefficients, we have

$$a^2\Sigma(\alpha-\beta)^2=18(b^2-ac),\quad a^4(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-27D,$$

$$a^3(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=-27(a^2d-3abc+2b^3),$$

by the help of which values, the expression obtained in the last article gives the relation between the covariants, due to Prof. Cayley,

$$J^2 - DU^2 = -4H^3.$$

This relation may also be easily verified by using the canonical form $U = ax^3 + dy^3$, in which case we have $D = a^2d^2$, $H = adxy$, $J = ad(ax^3 - dy^3)$. Any other relation between covariants may be similarly investigated. Thus we can prove that the discriminant of J is the cube of the discriminant of U, the former discriminant being for the canonical form a^6d^6. So again we see that the Hessian of J differs only by the factor D from the Hessian of U.

Prof. Cayley has used the relation just found between J, D, U, and H, to solve the cubic U, or, in other words, to resolve it into its linear factors. For, since $J^2 - DU^2$ is a perfect cube, we are led to infer that the factors $J \pm U\sqrt{D}$ will also be perfect cubes, and, in fact, the canonical form shows that they are $2a^2dx^3$ and $2ad^2y^3$. Now, since $xa^{\frac{1}{3}} + yd^{\frac{1}{3}}$ is one of the factors of the canonical form, it immediately follows that the factor in general is proportional to

$$(U\sqrt{D} + J)^{\frac{1}{3}} + (U\sqrt{D} - J)^{\frac{1}{3}},$$

a linear function which evidently vanishes on the supposition $U = 0$.

Ex. Let us take the same example as in Art. 166, $U = 4x^3 + 9x^2y + 18xy^2 + 17y^3$. Here we have $D = 1600$, $J = 110x^3 - 90x^2y - 630xy^2 - 670y^3$, whence

$$U\sqrt{D} + J = 10\,(3x + y)^3; \quad U\sqrt{D} - J = 50\,(x + 3y)^3;$$

and the factors are $3x + y + (x + 3y)\sqrt[3]{5}$.

197. The entire system of covariants for a cubic is also immediately found by Prof. Cayley's method explained Art. 192. We start with the three covariants U, H, J, whose leading coefficients are $L_1 = a$, $L_2 = ac - b^2$, $L_3 = a^2d - 3abc + 2b^3$. If we make $a = 0$, the last two become $-b^2$, $2b^3$, whence by eliminating b we have $4L_2^3 + L_3^2 = 0$. Thus we see that $4H^3 + J^2$ is divisible by a, and actually it is found to be divisible by a^2, the quotient being D or $a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$. We have thus obtained the new invariant D, together with the equation of

connection $4H^3+J^2=DU^2$. If in D we make $a=0$ it becomes $4db^3-3b^2c^2$, and since this combined with the preceding gives rise to no new relation between L_2, L_3, D, we learn that the system of covariants is complete.

198. *System of cubic and quadric.* Let these be

$$U=(a, b, c, d)(x, y)^3; \quad V=(A, B, C)(x, y)^2;$$

then the following is a list of the different independent covariants of the system. The figures added to each denote its order in the coefficients of the cubic and in those of the quadric.

Three cubic covariants, viz. the original cubic U, (1, 0); its cubicovariant (3, 0) which we call J, printed in full Arts. 195, 142, and the Jacobian of U, V, (1, 1) which is

$$(Ab-Ba)x^3+(2Ac-Bb-Ca)x^2y+(Ad+Bc-2Cb)xy^2+(Bd-Cc)y^3.$$

Three quadric covariants, viz. the original quadric V, (0, 1); the Hessian of the cubic (2, 0) and the Jacobian of these two (2, 1) which is

$$(A(ad-bc)-2B(ac-b^2),\ A(bd-c^2)-C(ac-b^2),$$
$$2B(bd-c^2)-C(ad-bc))(x, y)^2.$$

Four linear covariants, viz. L_1 (1, 1) which is obtained by substituting differential symbols in the quadric and operating on the cubic,

$$L_1=(aC-2bB+cA)x+(bC-2cB+dA)y;$$

L_2 (1, 2) which is obtained by operating in like manner with L_1 on the quadric,

$$L_2=\{aBC-b(2B^2+AC)+3cAB-dA^2\}x$$
$$+\{aC^2-3bBC+c(AC+2B^2)-dAB\}y,$$

and L_3 (3, 1), and L_4 (3, 2) which are obtained in like manner from the quadric and the cubicovariant J, and which may be written at length by substituting for a, b, &c. in the values of L_1, L_2 just given, the corresponding coefficients of J.

Five invariants, viz. Δ (0, 2) the discriminant of the quadric, D (4, 0) that of the cubic, I (2, 1) which is the intermediate invariant between the system of two quadrics V, H

$$I=A(bd-c^2)-B(ad-bc)+C(ac-b^2),$$

$R(2, 3)$, the resultant of the cubic and quadric, which formed by the methods of either Art. 67 or Art. 86, is

$$\begin{aligned} R = {} & a^2C^3 - 6abBC^2 + 6acC(2B^2 - AC) + ad(6ABC - 8B^3) \\ & + 9b^2AC^2 - 18bcABC + 6bdA(2B^2 - AC) \\ & + 9c^2A^2C - 6cdBA^2 + d^2A^3, \end{aligned}$$

and, finally, $M(4, 3)$, the resultant either of L_1, L_4, or of L_2, L_3 is

$$\begin{aligned} M = {} & a^3dC^3 - 3a^2bcC^3 - 6a^2bdBC^2 + 6a^2c^2BC^2 + 2ab^3C^3 + 6ab^2cBC^2 \\ & + 3ab^2dAC^2 + 12ab^2dB^2C - 6abc^2AC^2 - 24abc^2B^2C + 12ac^3ABC \\ & + 8ac^3B^3 - 3ac^2dA^2C - 12ac^2dAB^2 + 6acd^2A^2B - ad^3A^3 - 6b^4BC^2 \\ & + 3b^3cAC^2 + 12b^3cB^2C - 12b^3dABC - 8b^3dB^3 + 6b^2cdA^2C \\ & + 24b^2cdAB^2 - 6b^2d^2A^2B - 3bc^3A^2C - 12bc^3AB^2 - 6bc^2dA^2B \\ & + 3bcd^2A^3 + 6c^4A^2B - 2c^3dA^3. \end{aligned}$$

This last invariant M is a skew invariant (see Note, p. 131) and changes sign if we interchange x and y; the functions J, L_3, L_4 are also skew functions. In comparing different invariants we may conveniently make A and $C = 0$, which is equivalent to taking for x and y the two factors of the quadric. In this case the fundamental invariants are

$$\Delta = -B^2, \quad D = a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

$$I = -B(ad - bc), \quad R = -8B^3ad, \quad M = 8B^3(ac^3 - db^3).$$

Thus we have in the same case

$$L_1 = -2B(bx + cy), \quad L_2 = -2B^2(bx - cy),$$

and L the resultant of these two is $-8B^2bc$, whence we see immediately that L can be expressed in terms of the fundamental invariants; in fact, $L = R + 8\Delta I$. So, again, we see that the square of M can be expressed in terms of the other invariants, giving a relation between them. For we have

$$8(ac^3 + db^3) = 2(D - a^2d^2 + 3b^2c^2 + 6abcd),$$

whence $M^2 = 4B^6(D - a^2d^2 + 3b^2c^2 + 6abcd)^2 - 256B^6adb^3c^3$,

and if in this equation we substitute for ad, $\frac{-R}{8B^3}$, for bc, $-\frac{R + 8\Delta I}{8B^3}$,

and for B^2, $-\Delta$, we have the required relation

$$M^2 = -4\Delta^3 D^2 + D(R^2 + 12R\Delta I + 24\Delta^2 I^2) - 4RI^3 - 36\Delta I^4.$$

Geometrically, see Art. 190, if the cubic U be represented by three points on a conic its J covariant determines on the same conic the harmonic conjugates of each of the three with respect to the other two; the H covariant determines the double points of the involution of these six points. Or we may state it thus (see *Conics*, p. 387), the triangle touching the conic at the vertices of U is in perspective with U, the lines connecting corresponding vertices mark off J on the conic and intersect in the centre of perspective H_1, the axis of the perspective meets the conic in the points H: H_1 is also the pole of H with respect to the conic. Any quadratic V gives a right line meeting the conic in two points, and the line joining its pole to the centre of perspective H_1 is the Jacobian of H and V and is the axis of a new perspective whose centre is on the conic and given by the linear covariant L_1.

The line joining L_1 and the pole of H meets the conic again in L_3. L_2 is the harmonic of L_1 with respect to V, and L_4 is the point where H_1L_2 meets the conic again. The invariant I vanishes for any right line which passes through H_1.

199. *The quartic.* We come next to the quartic, which, as we have seen, pp. 128–9, has the two invariants

$$S = ae - 4bd + 3c^2 \quad \text{and} \quad T = ace + 2bcd - ad^2 - eb^2 - c^3.$$

We have shown (Art. 172) that the quartic may be reduced to the canonical form $x^4 + 6mx^2y^2 + y^4$, and for this form these invariants are $S = 1 + 3m^2$, $T = m - m^3$.

These invariants, expressed as symmetric functions of the roots, are $24S = a^2\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$, or

$$12S = a^2\{\beta\gamma + \alpha\delta + \omega(\gamma\alpha + \beta\delta) + \omega^2(\alpha\beta + \gamma\delta)\}\{\beta\gamma + \alpha\delta + \omega^2(\gamma\alpha + \beta\delta) + \omega(\alpha\beta + \gamma\delta)\},$$

also

$$\frac{72S}{a^2} = \{(\alpha-\beta)(\gamma-\delta) - (\alpha-\gamma)(\delta-\beta)\}^2 + \{(\alpha-\gamma)(\delta-\beta) - (\alpha-\delta)(\beta-\gamma)\}^2 + \{(\alpha-\delta)(\beta-\gamma) - (\alpha-\beta)(\gamma-\delta)\}^2,$$

and $432T = a^3\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)$, or, more conveniently,

$$432T = a^3\{(\alpha-\beta)(\gamma-\delta) - (\alpha-\gamma)(\beta-\delta)\}\{(\alpha-\gamma)(\delta-\beta) - (\alpha-\delta)(\beta-\gamma)\}\{(\alpha-\delta)(\beta-\gamma) - (\alpha-\beta)(\gamma-\delta)\}.$$

In the latter form it is easy to see that $T=0$ is the condition that the four points represented by the quartic should form a harmonic system, thus T may be called the harmonic invariant of the quartic, and in like manner S its equi-anharmonic invariant, see Note, p. 184. It was stated (Art. 171) that $T=0$ is the condition that the quartic can be reduced to the form x^4+y^4,* and that T can be expressed as a determinant

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

If Δ be the modulus of transformation, then (Art. 122) S and T become by transformation $\Delta^4 S$, $\Delta^6 T$, respectively; and the ratio $S^3 : T^2$ is *absolutely* unaltered by transformation.

200. *To express the discriminant in terms of S and T.* It has been already remarked (Art. 111) that the discriminant of a quantic must vanish, if the first two coefficients a and b vanish; for, in that case, the quantic, being divisible by y^2, has a square factor. On the other hand it is also true, that any invariant which vanishes when a and b are made $=0$, must contain the discriminant as a factor. Such an invariant, in fact, would vanish whenever the quantic had any square factor $(x-\alpha y)^2$; for, by linear transformation, the quantic could be brought to a form in which this factor was taken for y, and in which therefore the coefficients a and $b=0$. But an invariant which vanishes whenever any two roots of the quantic are equal, must, when expressed in terms of the roots, contain as a factor the difference between every two roots; that is to say, must contain the discriminant as a factor.

It is easy now, by means of S and T, to construct an invariant which shall vanish when we make a and $b=0$. For on this supposition S becomes $3c^2$, and T becomes $-c^3$; therefore S^3-27T^2 vanishes. Now this invariant of the sixth order in the coefficients is of the same order as that which we know

* Dr. Sylvester gives the name *catalecticant* to the invariant, which expresses that a quantic of order $2n$ can be reduced to the sum of n powers of the degree $2n$.

(Art. 105) the discriminant to be. It must therefore be the discriminant itself, and not the product of the discriminant by any other invariant. The discriminant is therefore

$$(ae - 4bd + 3c^2)^3 - 27\,(ace + 2bcd - ad^2 - eb^2 - c^3)^2.$$

We can in various ways verify this result. For instance, it appears from Art. 185,* that the discriminant of the canonical form $x^4 + 6mx^2y^2 + y^4$ is the square of the discriminant of the quadratic $x^2 + 6mxy + y^2$; that is to say, is $(1 - 9m^2)^2$. But

$$(1 - 9m^2)^2 = (1 + 3m^2)^3 - 27\,(m - m^3)^2.$$

We should also be led to the same form for the discriminant, by writing the quartic under a form more general than the canonical form, viz. $Ax^4 + By^4 + Cz^4$, where $x + y + z = 0$. In this case we have $a = A + C$, $e = B + C$, $b = c = d = C$, and we easily calculate $S = BC + CA + AB$, $T = ABC$. But if we equate to nothing the two differentials, viz. $Ax^3 - Cz^3$, $By^3 - Cz^3$, we get x^3, y^3, z^3 respectively proportional to BC, CA, AB; and, substituting in $x + y + z = 0$, we get the discriminant in the form

$$(BC)^{\frac{1}{3}} + (CA)^{\frac{1}{3}} + (AB)^{\frac{1}{3}} = 0,$$

which is $(BC + CA + AB)^3 - 27A^2B^2C^2 = 0$ or $S^3 - 27T^2 = 0$.

201. From the expression just given for the discriminant of a quartic in terms of S and T can be derived the relation (Art. 196) which connects the covariants of a cubic.

If we multiply two quantics together, the invariants of the compound quantic will be invariants of the system formed by the two components. If then we multiply a quantic by $x\xi + y\eta$, the invariants of the compound will (Art. 134) be contravariants of the original quantic; and when we change ξ and η into y and $-x$, will be covariants of it. If we apply this process to a cubic, the coefficients of the quartic so formed will be

$$ay,\ \tfrac{1}{4}(3by - ax)\ \tfrac{1}{2}(cy - bx),\ \tfrac{1}{4}(dy - 3cx),\ -dx;$$

* We may also see this directly, thus: The resultant of $ax^k + by^k$, $a'x^k + b'y^k$ is the k^{th} power of $ab' - ba'$, since the substitution of each root of the first equation in the second gives $ab' - ba'$. Now the discriminant of $ax^4 + 6cx^2y^2 + ey^4$ is the resultant of $ax^3 + 3cxy^2$, $3cx^2y + ey^3$. If we substitute $x = 0$ in the second, and $y = 0$ in the first, we get results e, a, respectively, and the resultant of $ax^2 + 3cy^2$, $3cx^2 + ey^2$ is $(ae - 9c^2)^2$. The discriminant is therefore $ae\,(ae - 9c^2)^2$.

and the invariants S and T of this quartic are found to be the covariants $-\frac{3}{4}H$, $\frac{1}{16}J$ of the cubic. But the discriminant of the product of any quantic by $x\xi + y\eta$, by Art. 110, becomes, when treated thus, the discriminant of U, multiplied by U^2. Expressing then the discriminant of the compound quartic in terms of its S and T, we get the relation connecting the H, J, and discriminant of the cubic.

202. A quartic has two covariants, viz. the Hessian H, whose leading coefficient is $ac - b^2$, and J the Jacobian of the quartic and its Hessian, whose leader is $a^2d - 3abc + 2b^3$.

The Hessian is the evectant of T, its value is

$$H = (ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3y + (ae + 2bd - 3c^2)\,x^2y^2 + 2\,(be - cd)\,xy^3 + (ce - d^2)\,y^4$$

$$= -\begin{vmatrix} 0, & y^2, & -xy, & x^2 \\ y^2, & a, & b, & c \\ -xy, & b, & c, & d \\ x^2, & c, & d, & e \end{vmatrix},$$

$$\text{or}\quad 3H = \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ 3y^2, & -2xy, & x^2, & 0 \\ 0, & y^2, & -2xy, & 3x^2 \end{vmatrix} = \begin{vmatrix} d, & -3c, & 3b, & -a \\ e, & -3d, & 3c, & -b \\ x^2, & 2xy, & y^2, & 0 \\ 0, & x^2, & 2xy, & y^2 \end{vmatrix},$$

and becomes, for the canonical form,

$$m\,(x^4 + y^4) + (1 - 3m^2)\,x^2y^2.$$

Expressed in terms of the roots, it is

$$-48H = a^2\Sigma\,(\alpha - \beta)^2\,(x - \gamma)^2\,(x - \delta)^2.$$

The covariant J, which symbolically is $\overline{12^2}\ \overline{13}$, written at length, is

$$J = (a^2d - 3abc + 2b^3,\ a^2e + 2abd - 9ac^2 + 6b^2c,\ 5abe - 15acd + 10b^2d,$$
$$-10ad^2 + 10b^2e,\ -5ade + 15bce - 10bd^2,$$
$$-ae^2 - 2bde + 9c^2e - 6cd^2,\ -be^2 - 2d^3 + 3cde)(x, y)^6$$

$$= \frac{1}{3}\begin{vmatrix} x^3, & 3x^2y, & 3xy^2, & y^3, & 0 \\ 0, & x^3, & 3x^2y, & 3xy^2, & y^3 \\ -d, & 3c, & -3b, & a, & 0 \\ -e, & 2d, & 0, & -2b, & a \\ 0, & -e, & 3d, & -3c, & b \end{vmatrix},$$

and for the canonical form $(1 - 9m^2)\,xy\,(x^4 - y^4)$.

We have just seen that the x and y of the canonical form which we use are factors of J, but it will be remembered (Art. 172) that the problem of reducing a quartic to its canonical form depends on the solution of a cubic equation; hence, the factors of J are the x and y of the three canonical forms. This may be connected with the theory explained in Art. 177. If U and V are any two quartics, six values of λ can be found, such that $U+\lambda V$ shall have a square factor, and those six factors are the factors of the Jacobian of U and V. But when V is the Hessian of U, the sextic in question becomes a perfect square, and there are three values of λ, for each of which $U+\lambda V$ contains *two* square factors, but these factors are still the factors of the Jacobian of U and V. The geometrical meaning of J may be stated as follows: let the quartic represent four points on a line A, B, C, D, then these determine three different systems in involution (according as B, C or D is taken as the conjugate of A), and the foci of these three systems are given by the covariant J.

From the last remark we can express the factors of J in terms of the roots. In fact, by Ex. 7, p. 25, the double points of the involution formed by β, γ; α, δ are determined by the quadratic $\begin{vmatrix} x^2, & 2x, & 1 \\ \beta\gamma, & \beta+\gamma, & 1 \\ \alpha\delta, & \alpha+\delta, & 1 \end{vmatrix} = 0$. Similar equations determine the foci for the other two systems.

Ex. 1. To break up the quartic into two quadratic factors.

Let $(px^2+2qxy+ry^2)(p'x^2+2q'xy+r'y^2)$ be identified with the quartic, and substitute in their places for $pp'=a$, $pq'+qp'=2b$, $qr'+rq'=2d$, $rr'=e$, $pr'+rp'=2(c+2\rho)$, $qq'=c-\rho$ in the identity $\begin{vmatrix} p, & p' \\ q, & q' \\ r, & r' \end{vmatrix} \begin{vmatrix} p', & p \\ q', & q \\ r', & r \end{vmatrix} = 0$, expanded as in Art. 25. The reducing cubic is found to be $\begin{vmatrix} a, & b, & c+2\rho \\ b, & c-\rho, & d \\ c+2\rho, & d, & e \end{vmatrix} = 0$ or $4\rho^3 - S\rho + T = 0$.

Now, when we write

$$(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)=12\rho_1,$$
$$(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)=12\rho_2,$$
$$(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)=12\rho_3$$

in the expressions for S and T in terms of the roots, Art. 199, they become

$$S=-4a^2(\rho_2\rho_3+\rho_3\rho_1+\rho_1\rho_2),$$
$$T=-4a^3\rho_1\rho_2\rho_3.$$

Hence, since $\rho_1+\rho_2+\rho_3=0$; $a\rho_1$, $a\rho_2$, $a\rho_3$ are the roots of the cubic

$$4\rho^3 - S\rho + T = 0.$$

Ex. 2. To discuss the relations between the quadratic factors of J.

Writing out the determinant forms, let us call

$$u=(\beta+\gamma-\alpha-\delta)\,x^2-2\,(\beta\gamma-\alpha\delta)\,xy+[\beta\gamma\,(\alpha+\delta)-\alpha\delta\,(\beta+\gamma)]\,y^2=a_1x^2+2b_1xy+c_1y^2,$$

$$v=(\gamma+\alpha-\beta-\delta)\,x^2-2\,(\gamma\alpha-\beta\delta)\,xy+[\gamma\alpha\,(\beta+\delta)-\beta\delta\,(\gamma+\alpha)]\,y^2=a_2x^2+2b_2xy+c_2y^2,$$

$$w=(\alpha+\beta-\gamma-\delta)\,x^2-2\,(\alpha\beta-\gamma\delta)\,xy+[\alpha\beta\,(\gamma+\delta)-\gamma\delta\,(\alpha+\beta)]\,y^2=a_3x^2+2b_3xy+c_3y^2.$$

We have thus

$$v-w=-2(\beta-\gamma)(x-\alpha y)(x-\delta y),\quad w-u=-2(\gamma-\alpha)(x-\beta\gamma)(x-\delta\gamma),\quad u-v=-2(\alpha-\beta)(x-\gamma y)(x-\delta\gamma),$$

$$v+w=2(\alpha-\delta)(x-\beta y)(x-\gamma y),\quad w+u=2(\beta-\delta)(x-\gamma y)(x-\alpha y),\quad u+v=2(\gamma-\delta)(x-\alpha\gamma)(x-\beta\gamma).$$

Hence
$$\frac{v^2-w^2}{\rho_2-\rho_3}=\frac{w^2-u^2}{\rho_3-\rho_1}=\frac{u^2-v^2}{\rho_1-\rho_2}=\frac{16U}{a}.$$

Thus it appears at once that the identical relation (compare p. 181) between u, v, w is $\rho_1u^2+\rho_2v^2+\rho_3w^2=0$. Hence, as this relation involves only the squares, the quadratics are harmonic in pairs. The same thing is found by actual calculation:

$$a_1c_2+c_1a_2-2b_1b_2=0,\ \&c.,$$

also $a_1c_1-b_1^2=(\gamma-\alpha)\,(\beta-\delta)\,(\alpha-\beta)\,(\gamma-\delta)=16\,(\rho_3-\rho_1)\,(\rho_1-\rho_2)$, &c.,

or, writing $\Delta=(\rho_2-\rho_3)\,(\rho_3-\rho_1)\,(\rho_1-\rho_2)$, $(a_1c_1-b_1^2)\,(\rho_2-\rho_3)=\&c.=16\Delta$.

The value already given for H in terms of the roots may be written

$$a^2\,(u^2+v^2+w^2)=-48H.$$

Combining this with the values of U given above we get

$$16aU=\frac{a^2u^2+16H}{\rho_1}=\frac{a^2v^2+16H}{\rho_2}=\frac{a^2w^2+16H}{\rho_3}.$$

Ex. 3. We have seen that J can differ only by a numerical factor from the product of the three quadratics u, v, w. To determine it we may compare the leading terms of the two forms, or, expressing the symmetric function in terms of the coefficients, find that

$$a^3\,(\beta+\gamma-\alpha-\delta)\,(\gamma+\alpha-\beta-\delta)\,(\alpha+\beta-\gamma-\delta)=32\,(a^2d-3abc+2b^3),$$

whence
$$a^3uvw=32J.$$

203. *Solution of the quartic.* This is the same problem as that of the reduction of the quartic to its canonical form $ax^4+6cx^2y^2+ey^4$, for in this form it can be solved like a quadratic. One method of reduction has been explained (Art. 172); the reduction may also be effected by means of the values given for S and T. Imagine the variables transformed by a linear transformation whose modulus is unity, and so that the new b and d shall vanish; then we have $S=ae+3c^2$, $T=ace-c^3$; and the new c is given by the equation $4c^3-Sc+T=0$. We get the x and y which occur in the canonical form from the equations

$$U=ax^4+6cx^2y^2+ey^4,\quad H=acx^4+(ae-3c^2)\,x^2y^2+cey^4,$$

whence
$$cU-H=(9c^2-ae)\,x^2y^2.$$

Our process then is to solve for c from the cubic just given, and with one of the values of c to form $cU-H$ which will be found to be a perfect square. Taking the square root and breaking it up into its factors we find the new x and y, and consequently know the transformation, by means of which the given quartic can be brought to the canonical form. Having got it to the form $ax^4+6cx^2y^2+ey^4$, we can of course, if we please, make the coefficients of x^4 and y^4 unity, by writing x^2 and y^2 for $x^2\sqrt{(a)}$, and $y^2\sqrt{(e)}$.

Ex. Solve the equation

$$x^4+8x^3y-12x^2y^2+104xy^3-20y^4=0.$$

We have here $S=-216$, $T=-756$, and our cubic is $4c^3+216c=756$, of which $c=3$ is a root. The Hessian is

$$H=-6x^4+60x^3y+72x^2y^2+24xy^3-636y^4,$$

$$3U-H=9\,(x^4-4x^3y-12x^2y^2+32xy^3+64y^4)=9\,(x^2-2xy-8y^2)^2.$$

The variables then of the canonical form are $X=x+2y$, $Y=x-4y$, which give $6x=4X+2Y$, $6y=X-Y$; whence, substituting in the given quartic, the canonical form is found to be $3X^4+2X^2Y^2-Y^4$. The roots then are given by the equations

$$(x+2y)\sqrt{(3)}=x-4y,\quad (x+2y)\sqrt{(-1)}=x-4y.$$

204. Since J is proportional to the continued product of the x and y of the three canonical forms, and since we have just seen that the square of the product of one set of x and y is $cU-H$, where c is one of the roots of the cubic $4c^3-Sc+T=0$, we have J^2 proportional to $4H^3-SHU^2+TU^3$. By calculating with the canonical form, we find the actual value to be $-J^2$. Or, again, we saw in Ex. 2, Art. 202, that

$$16\,(a\rho_1U-H)=a^2u^2,$$

$$16\,(a\rho_2U-H)=a^2v^2,$$

$$16\,(a\rho_3U-H)=a^2w^2,$$

and in the following example that

$$a^3uvw=32J,$$

hence by the values of ρ_1, ρ_2, ρ_3 of Ex. 1 of same article

$$J^2=-4H^3+U^2HS-U^3T.$$

205. Prof. Cayley has given the root of the quartic in a more symmetrical form. It has been shown that $a\rho_1U-H$, $a\rho_2U-H$, $a\rho_3U-H$ are perfect squares severally of u, v, w.

If, further, we enquire under what conditions $\lambda u + \mu v + \nu w$ is a perfect square, we find that

$$\lambda^2 (a_1 c_1 - b_1^2) + \mu^2 (a_2 c_2 - b_2) + \nu^2 (a_3 c_3 - b_3^2) = 0$$

must be satisfied (compare Ex. 2, Art. 202), or, as it may be written,

$$\frac{\lambda^2}{\rho_2 - \rho_3} + \frac{\mu^2}{\rho_3 - \rho_1} + \frac{\nu^2}{\rho_1 - \rho_2} = 0.$$

If we further wish to make

$$\lambda \sqrt{(a\rho_1 U - H)} + \mu \sqrt{(a\rho_2 U - H)} + \nu \sqrt{(a\rho_3 U - H)}$$

vanish with U, we must have $\lambda + \mu + \nu = 0$, whence, solving, we find

$$\frac{\lambda}{\rho_2 - \rho_3} = \frac{\mu}{\rho_3 - \rho_1} = \frac{\nu}{\rho_1 - \rho_2},$$*

thus

$$(\rho_2 - \rho_3) \sqrt{(a\rho_1 U - H)} + (\rho_3 - \rho_1) \sqrt{(a\rho_2 U - H)} + (\rho_1 - \rho_2) \sqrt{(a\rho_3 U - H)}.$$

Ex. 1. This may be verified by means of the canonical form, taking for simplicity a and $e = 1$. If we solve the equation $4z^3 - z(1 + 3c^2) + c - c^3 = 0$, we find the three roots to be c, $-\frac{1}{2}(c+1)$, $-\frac{1}{2}(c-1)$; and the three corresponding values of $H - cU$ are

$$(1 - 9c^2) x^2 y^2, \quad \tfrac{1}{2}(3c+1)(x^2+y^2)^2, \quad \tfrac{1}{2}(3c-1)(x^2-y^2)^2.$$

Now in order that any quantity of the form

$$\alpha xy + \beta (x^2 + y^2) + \gamma (x^2 - y^2)$$

may be a perfect square, we must obviously have $\alpha^2 = 4(\beta^2 - \gamma^2)$, which is verified when

$$\alpha^2 = 1 - 9c^2, \quad \beta^2 = \tfrac{1}{8}(3c-1)^2(3c+1), \quad \gamma^2 = \tfrac{1}{8}(3c+1)^2(3c-1).$$

Ex. 2. If this method be applied to the example Art. 203, the other values of c are $\frac{1}{3}\{-3 \pm 9\sqrt{(-3)}\}$; and the squares of the linear factors of the quartic are given in the form

$$-2\sqrt{(3)}\{x^2 - 2xy - 8y^2\} \pm \tfrac{1}{2}\{1 - \sqrt{(-3)}\}[\{1 + \sqrt{(-3)}\}x^2 + \{10 - 2\sqrt{(-3)}\}xy - \{2 - 10\sqrt{(-3)}\}y^2]$$
$$\pm \tfrac{1}{2}\{1 + \sqrt{(-3)}\}[\{1 - \sqrt{(-3)}\}x^2 + \{10 + 2\sqrt{(-3)}\}xy - \{2 + 10\sqrt{(-3)}\}y^2].$$

Ex. 3. The factors of H are the values of

$$(\rho_2 - \rho_3)\sqrt{(a\rho_1^2 U - \rho_1 H)} + (\rho_3 - \rho_1)\sqrt{(a\rho_2^2 U - \rho_2 H)} + (\rho_1 - \rho_2)\sqrt{(a\rho_3^2 U - \rho_3 H)}.$$

206. It remains to distinguish the cases in which the transformation to the canonical form is made by a real or by an

* Burnside, *Hermathena* IV. 1876.

imaginary substitution. The discriminant of the canonical form is, as we have seen (note, Art. 200), $ae(ae-9c^2)^2$; and since the sign of the discriminant is unaffected by linear transformation, we see that whenever the discriminant is positive, a and e of the canonical form have like signs; and when the discriminant is negative, unlike signs. Now the form $ax^4+6cx^2y^2+ey^4$ evidently resolves itself into two factors of the form, either $(x^2+\lambda y^2)(x^2+\mu y^2)$ or $(x^2-\lambda y^2)(x^2-\mu y^2)$; that is to say, the quartic has either four imaginary roots or four real roots. On the contrary, if a and e have opposite signs, the two factors are of the form $(x^2+\lambda y^2)(x^2-\mu y^2)$, or the quartic has two real and two imaginary roots. Hence, then, when the discriminant is negative, that is to say, when S^3 is less than $27T^2$, the quartic has two real roots and two imaginary; and when the discriminant is positive, it has either four real or four imaginary roots.* Now the discriminant of the equation $4c^3-Sc+T=0$ is $27T^2-S^3$, therefore (Art. 167) when S^3 is less than $27T^2$, the equation in c has one root real and two imaginary; in the other case it has three real roots. Hence when a and e have opposite signs, that is, when the quartic has two real and two imaginary roots, the transformation can be effected in one way only. Next, if a and e have like signs, in which case the equation can be brought to the form $x^4+6mx^2y^2+y^4$, it is easy to see that the equation can by two other linear transformations be brought to the same form; for write $x+y$ and $x-y$ for x and y, and we have $(1+3m)x^4+6(1-m)x^2y^2+(1+3m)y^4$. Write $x+y\sqrt{(-1)}$, and $x-y\sqrt{(-1)}$ for x and y, and we have $(1+3m)x^4+6(m-1)x^2y^2+(1+3m)y^4$. Hence when a and e have the same sign, that is, when the quartic has four real or four imaginary roots, though there are three real values for c, one of these corresponds to imaginary values of x and y; and there are only two real ways of making the transformation.

* The signs of the invariants do not enable us to distinguish the case of four real roots from that of four imaginary; but the application of Sturm's theorem shews that (the discriminant being positive), when the roots are all real, both the quantities b^2-ac and $3aT+2(b^2-ac)S$ are positive, while if either is negative the four roots are imaginary. (Cayley, *Quarterly Journal*, vol. IV., p. 10).

The same thing may also be seen thus. Imagine the quartic to have been resolved into two real quadratic factors

$$(a, b, c\mathcal{X}x, y)^2, \quad (a', b', c'\mathcal{X}x, y)^2;$$

then these two factors U, V can, by simultaneous transformation, be brought to the forms AX^2+BY^2, $A'X^2+B'Y^2$, where X^2 and Y^2 are the values of $\lambda U+V$ corresponding to the two values of λ given by the equation

$$(ac-b^2)\lambda^2+(ac'+ca'-2bb')\lambda+(a'c'-b'^2)=0.$$

In order that the values of λ should be real, we must have the eliminant of the two quadratics positive, or

$$(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha')(\beta-\beta')$$

positive. Thus then when the quantic has four real roots, if we take for α and β the two greatest roots, and for α' and β' the two least; or, again, if we take for α and β the two extreme roots, and for α' and β' the two mean roots, we get real values for λ. In the remaining case we get imaginary values. If either of the quadratics has imaginary roots, the resultant of the two is positive, and the values of λ real.

207. *Conditions for two pairs of equal roots.* If any quantic have a square factor x^2, this will be also a factor in the Hessian. For the second differential U_{22} contains x^2, and U_{12} contains x, therefore x^2 will be a factor in $U_{11}U_{22}-U_{12}^2$. If then a quartic have two square factors, both will be factors in the Hessian, which, being of the fourth degree, can therefore differ only by a numerical factor from the quartic itself. In fact, if a quartic have two square factors, by taking these for x^2 and y^2, the quartic may be reduced to the form cx^2y^2; but, by making a, b, d, e all $=0$, the Hessian, as given Art. 202, reduces to $-3c^2x^2y^2$.

Thus then by expressing that a quartic differs only by a factor from its Hessian, we get the system of conditions that the quartic shall have two square factors, viz.

$$\frac{ac-b^2}{a}=\frac{ad-bc}{2b}=\frac{ae+2bd-3c^2}{6c}=\frac{be-cd}{2d}=\frac{ce-d^2}{e},$$

a system equivalent to two conditions, as may be verified in different ways.

We have, in Art. 138, given other ways of forming these conditions. From the expression (Art. 202) for the covariant J in terms of the roots, it appears that every term of it must vanish identically if any two pairs of roots become respectively equal. This also follows from the consideration that J is the Jacobian of the quartic and its Hessian, and must vanish identically when these two only differ by a factor; now the coefficients in J are, only in a different form, the conditions already written. Again, we have said (Art. 138) that in the same case the covariant $\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)^4$ vanishes identically. But this, it will be found, is the same as $3TU-2SH$; and we can easily verify that *this covariant vanishes when the quartic has two square factors;* for, making a, b, d, e all $=0$, U reduces to $6cx^2y^2$, H to $-3c^2x^2y^2$, T to $-c^3$, and S to $3c^2$. Thus, then, we see that in the system of conditions given above, the common value of the fractions is $\dfrac{3T}{2S}$.

208. We next show by Prof. Cayley's method (Art. 194) that the system of invariants and covariants already given is *complete*. We start with the seminvariants a, $ac-b^2$, $a^2d-3abc+2b^3$, $a^3e-4a^2bd+6ab^2c-3b^4$, the first three being the leading coefficients of U, H, J. Since any relation between the leading coefficients of covariants implies a similar relation between the covariants themselves, there will be no inconvenience in calling the first three terms by the names U, H, J; the fourth we shall call provisionally L. If now we make $a=0$, we have $U'=0$, $H'=-b^2$, $J'=2b^3$, $L'=-3b^4$; and by eliminating b between the second and third, and second and fourth of these equations, we have

$$4H'^3+J'^2=0, \quad 3H'^2+L'=0.*$$

Now these two quantities which vanish on the supposition $a=0$,

* It is easy to see that any result of elimination, obtained by combining these equations differently, will vanish when the two equations, written above, are satisfied.

will, when we give H, J, L their general values, be divisible by a power of a. The first has been already discussed in the theory of the cubic. It gives $4H^3 + J^2 = U^2 D$, where

$$D = a^2d^2 + 4ac^3 - 6abcd + 4b^3d - 3b^2c^2.$$

The second, treated in like manner, gives $3H^2 + L = U^2 S$. We have thus been led to the two new seminvariants D, S, and we may dismiss L, which we have seen can be expressed as a function of simpler covariants. Making $a = 0$ again in D and S, we have

$$D' = b^2(4bd - 3c^2), \quad S' = (-4bd + 3c^2),$$

whence, since $H' = -b^2$, we have $D' - H'S' = 0$. And giving D, H, S their general values, we find $D - HS = -UT$. We are thus led to the new invariant T and may dismiss D, which has been linearly expressed in terms of simpler covariants. Making $a = 0$ in T, we have $T' = 2bcd - eb^2 - c^3$, and we cannot now by elimination of b, c, d, e obtain any new relation between H', J', S', T'. The system is therefore complete, consisting of U, H, J, S, T with the equation of connection

$$4H^3 + J^2 = U^2(HS - UT').$$

209. We have already (Art. 190) mentioned Mr. Burnside's remark on the identity of the theory of the quartic with that of a pair of conics (*Conics*, Art. 370). By the substitution x, y, z for x^2, $2xy$, y^2, the quartic becomes

$$ax^2 + cy^2 + ez^2 + 2dyz + 2czx + 2bxy = 0,$$

with the identity $4xz - y^2 = 0$. Calling these two conics u and v the discriminant of $u + \lambda v$ is $4\lambda^3 - S\lambda + T = 0$. Thus we see that the invariants of the system of two conics are also invariants of the quartic. The solution of the quartic evidently is given by the cubic in λ just written; for if λ be one of its roots, we know that the ternary quadratic is resolvable into two factors. The discriminant of the resolving cubic, $27T^2 - S^3$, which vanishes when two conics touch, gives also the condition that the quartics should have equal roots. To the Hessian of the quartic answers the harmonic conic (*Conics*, Art. 378) of the system of two conics, and to the sextic covariant J, which is the Jacobian of the quartic and its Hessian, answers the Jacobian

of the two conics and their harmonic conic; that is to say, the sides of the self-conjugate triangle common to the two conics. The expression for the square of J in terms of U, H, S, T, answers to the expression given, *Conics*, Art. 388*a*.

210. Since H is a covàriant of U, it follows that if α and β be any constants, $\alpha U + 6\beta H$ will be a covariant of U, whose invariants also will be invariants of U. The following are the values of the S, T, and discriminant R, of this form:

$$S(\alpha U + 6\beta H) = S\alpha^2 + 18T\alpha\beta + 3S^2\beta^2,$$
$$T(\alpha U + 6\beta H) = T\alpha^3 + S^2\alpha^2\beta + 9ST\alpha\beta^2 + (54T^2 - S^3)\beta^3,$$
$$R(\alpha U + 6\beta H) = R(\alpha^3 - 9S\alpha\beta^2 - 54T\beta^3)^2.$$

The last is a perfect square, because, as was already mentioned, instead of six cases where $\alpha U + 6\beta H$ has a square factor, we have three cases where it has two square factors.

Hermite has noticed that if we call G the function of α, β, $\alpha^3 - 9S\alpha\beta^2 - 54T\beta^3$, then the values just given for the S and T of $\alpha U + 6\beta H$ are respectively the Hessian and the cubicovariant of G. The discriminant of G differs only by a numerical factor from the discriminant of U.

The covariants of $\alpha U + 6\beta H$ are also covariants of U. Its Hessian is

$$(\alpha\beta S + 9\beta^2 T)\ U + (\alpha^2 - 3\beta^2 S)\ H,$$

which is the Jacobian, with respect to α, β, of G and $\alpha U + 6\beta H$. Since J is a combinant of the system U, H, the J of $\alpha U + 6\beta H$ will be the same, multiplied, however, by the numerical factor G. The Hessian of J is $S^2U^2 - 36TUH + 12SH^2$, which is the resultant of $\alpha U + 6\beta H$ and the Hessian of G. Prof. Cayley has thrown this into the form

$$\left(SU - \frac{18T}{S}H\right)^2 + \frac{12}{S^2}(S^3 - 27T^2)H^2,$$

shewing that it is a perfect square when the discriminant of U vanishes.

Ex. 1. For the form $\alpha U - \beta H$ the function G of α, β, is $4\alpha^3 - S\alpha\beta^2 + T\beta^3$.

Thus the quartics of the system $\alpha U - \beta H$ whose discriminant vanishes are determined by the reducing cubic of U. The same cubic determines what quartic of the system coincides with its Hessian.

Ex. 2. The factors of $\alpha U - \beta H$ are, as in Art. 205, the values of

$$(\rho_2 - \rho_3)\sqrt{(\beta\alpha\rho_1 - \alpha)}\sqrt{(\alpha\rho_1 U - H)} + (\rho_3 - \rho_1)\sqrt{(\beta\alpha\rho_2 - \alpha)}\sqrt{(\alpha\rho_2 U - H)}$$
$$+ (\rho_1 - \rho_2)\sqrt{(\beta\alpha\rho_3 - \alpha)}\sqrt{(\alpha\rho_3 U - H)}.$$

211. Since it has been just proved that the Hessian of the Hessian of a quartic is of the form $\alpha TU + \beta SH$, we can infer, as in Art. 194, that the same is true of the Hessian of the Hessian of any quantic. For if we form the Hessian of $u_{11}u_{22} - u_{12}^2$, this involves the second, third, and fourth differentials of u. But, by the equations $(n-3)\,u_{111} = xu_{1111} + yu_{1112}$, &c., we can express the second and third differentials in terms of the fourth, and so write the second Hessian as a function of the fourth differentials only, and of the x and y which we have introduced, and which, it will be found, enter in the fourth degree. It will then be a covariant of the quartic emanant. Now every covariant of a quartic is a function of U and H (Art. 208), and when the covariant is of the fourth degree it must be a linear function of these quantities. Actually it is found to be proportional to $(2n-5)\,TU - SH$, where S and T are invariants of the quartic emanant and, as in Art. 141, covariants of the higher quantic.

212. *System of a biquadratic and quadratic.* This system is most easily dealt with by Mr. Burnside's method, Art. 190. Let the quadratic be $\alpha x^2 + 2\beta xy + \gamma y^2$, and let the quartic be given by the general equation, then (Art. 190) this is equivalent to the system of two conics and a right line

$$ax^2 + cy^2 + ez^2 + 2dyz + 2czx + 2bxy,\ 4xz - y^2,\ \alpha x + \beta y + \gamma z,$$

the properties of which have been discussed, *Conics*, Art. 370, &c. For example, the formula of *Conics*, Art. 377, expressing the resultant of the three ternary quantics in terms of simpler invariants, gives at the same time an expression for the resultant of the two binary quantics. The formula just cited gives the resultant as $\phi^2 - 4\Sigma\Sigma'$, where

$$\Sigma = \alpha^2(ce - d^2) + \beta^2(ae - c^2) + \gamma^2(ac - b^2) + 2(bc - ad)\beta\gamma + 2(bd - c^2)\gamma\alpha + 2(cd - be)\alpha\beta,$$

$$\Sigma' = 4(\alpha\gamma - \beta^2),$$

$$\phi = e\alpha^2 + 4c\beta^2 + a\gamma^2 - 4b\beta\gamma + 2c\gamma\alpha - 4d\alpha\beta.$$

In the above, Σ' is proportional to the discriminant of the quadratic, ϕ is an invariant got by substituting differential symbols in the quadratic, squaring, and operating on the quartic;

if we operate in like manner on the Hessian of the quartic, we get an invariant of the same order as Σ, but differing from it by a multiple of $S\Sigma'$. If we treat Σ, Σ', ϕ as conics and form their Jacobian, we get another invariant of the system of conics, the vanishing of which geometrically represents the condition that the right line shall pass through one of the vertices of the common self-conjugate triangle of the two conics. It is

$$\alpha^3(3cde - 2d^3 - be^2) + \alpha^2\gamma(3bce - 2bd^2 - ade)$$
$$+ \alpha\gamma^2(abe + 2b^2d - 3acd) + \gamma^3(a^2d + 2b^3 - 3abc)$$
$$+ \beta\alpha^2(6cd^2 + 2bde - 9c^2e + ae^2) + \alpha\beta\gamma(6ad^2 - 6b^2e)$$
$$+ \beta\gamma^2(-a^2e - 6b^2c + 9ac^2 - 2abd) + \beta^2\alpha(12bce - 8bd^2 - 4ade)$$
$$+ \beta^2\gamma(-12acd + 8b^2d + 4abe) + \beta^3(4ad^2 - 4b^2e).$$

This is a skew invariant of the binary system of the quartic and quadratic. The formula (*Conics*, Art. 388*a*) gives an expression for the square of this in terms of the other invariants. From what has been stated, as to the geometric meaning of the skew-invariant, it follows that if it vanishes two of the right lines which pass through the intersections of the two conics intersect on the given line; that is to say, these equations are of the form $L \pm M = 0$, where L is the given line and M some other line. The corresponding property for the binary equations is, that the vanishing of the skew-invariant is the condition that the given quartic can be resolved into two quadratic factors $L \pm M$ where L is the given and M some other quadratic. The system of quartic and quadratic has only these six independent invariants now indicated; viz. the S and T of the quartic, the discriminant of the quadratic, those which we have just called Σ and ϕ, and the skew-invariant.

We get immediately two quadratic covariants of the binary system by introducing differential symbols into the given quadratic, and operating on the quartic and its Hessian. Thus we get the two forms

$$(c\alpha - 2b\beta + a\gamma)x^2 + 2(d\alpha - 2c\beta + b\gamma)xy + (e\alpha - 2d\beta + c\gamma)y^2;$$
$$\{\alpha(ae + 2bd - 3c^2) - 6\beta(ad - bc) + 6\gamma(ac - b^2)\}x^2$$
$$+ \{6\alpha(be - cd) - 4\beta(ae + 2bd - 3c^2) + 6\gamma(ad - bc)\}xy$$
$$+ \{6\alpha(ce - d^2) - 6\beta(be - cd) + \gamma(ae + 2bd - 3c^2)\}y^2.$$

To these two binary covariants answer covariant right lines in the ternary system, which are found by taking the pole of the given line with regard to either conic and then the polar of this point with regard to the other. Having now three quadratics, viz. the given one and the two just found, we obtain, three more quadratic covariants by taking the intermediate covariant (Art. 193) of each pair; and these six quadratics complete the system of quadratic covariants. There are five quartic covariants, viz. in addition to the given quartic and its Hessian, the Jacobian of the quartic and quadratic, of the Hessian and quadratic, and of the Hessian and the first covariant quadratic. Lastly, there is the sextic covariant of the quartic. The eighteen forms enumerated make up the complete system.

Ex. 1. Required the right lines which have the same pole with respect to each conic.

The first of the above quadratic covariants identified with $ax^2 + 2\beta xy + \gamma y^2$ gives

$$a\gamma - 2b\beta + c\alpha = \lambda\alpha,$$
$$b\gamma - 2c\beta + d\alpha = \lambda\beta,$$
$$c\gamma - 2d\beta + e\alpha = \lambda\gamma;$$

whence, eliminating α, β, γ,

$$\begin{vmatrix} a, & b, & c-\lambda \\ b, & c+\frac{1}{2}\lambda, & d \\ c-\lambda, & d, & e \end{vmatrix} = 0,$$

the reducing cubic of the biquadratic (see Art. 172) is thus found: hence there are three such lines, and the reduction of a quartic to its canonical form is again seen to be the same problem as that of two conics to a self conjugate triangle.

Ex. 2. The above skew invariant is also found by operating with the sextic covariant of the biquadratic on the cube of the quadratic. Geometrically, we may determine it, either by expressing that the quadratic is a pair of a system in involution with the biquadratic, or by expressing that the quadratic is harmonic with one of the factors of the covariant sextic (Art. 202, Ex. 1).

213. *System of two cubics.* We begin with those invariants of the system of two cubics $(a, b, c, d\chi x, y)^3$, $(a', b', c', d'\chi x, y)^3$, which are also combinants. The simplest is (see Art. 140, Ex. 2) $(ad') - 3(bc')$, which we shall refer to as the invariant P. The properties of this system may be studied most conveniently by throwing the equations into the form

$$Au^3 + Bv^3 + Cw^3,\ A'u^3 + B'v^3 + C'w^3,$$

a form to which the two cubics can be reduced in an infinity of ways. For, the cubics contain four constants each, or eight

in all. And the form just written contains six constants explicitly; and u, v, w contain implicitly a constant each, since u stands for $x + \lambda y$, &c. The second form then is equivalent to one with nine constants, that is to say, one constant more than is necessary to enable us to identify it with the general form.

Any three binary quantics of the first degree are obviously connected by an identical relation of the form $au + \beta v + \gamma w = 0$. We write x, y, z for au, βv, γw, so that the two cubics are $Ax^3 + By^3 + Cz^3$, $A'x^3 + B'y^3 + C'z^3$, where $x + y + z = 0$.

Putting for z its value, and writing the cubics

$$(A - C, -C, -C, B - C)(x, y)^3, (A' - C', -C', -C', B' - C')(x, y)^3;$$

then forming the invariant P of the system, we find it to be

$$(BC') + (CA') + (AB').$$

The resultant of the system is found by solving between the equations $Ax^3 + By^3 + Cz^3 = 0$, $A'x^3 + B'y^3 + C'z^3 = 0$, whence we get $x^3 = (BC')$, $y^3 = (CA')$, $z^3 = (AB')$; substituting in the identity $x + y + z = 0$, the resultant is

$$(BC')^{\frac{1}{3}} + (CA')^{\frac{1}{3}} + (AB')^{\frac{1}{3}} = 0,$$

or
$$\{(BC') + (CA') + (AB')\}^3 = 27\,(BC')\,(CA')\,(AB').$$

Now, if we denote the two cubics by u and v, it has been proved, Art. 180, that there is an invariant, which we shall call Q, of the third order in the coefficients of each cubic, which expresses the condition of its being possible to determine λ, so that $u + \lambda v$ shall be a perfect cube. This invariant is identical with the product $(BC')\,(CA')\,(AB')$, which is of the same degree in the coefficients. For, if any factor (AB') in this product vanish, $Av - A'u$ evidently reduces to the perfect cube $(AC')\,z^3$. It follows then that the resultant is of the form $P^3 - 27\,Q$.

214. If it were required to form directly the invariant Q for the form $(a, b, c, d)(x, y)^3$, $(a', b', c', d')(x, y)^3$, we might proceed as follows. If $u + \lambda v$ be a perfect cube, its three second differentials will simultaneously vanish; or, for proper values of

x, y, λ, we have

$$ax + by + \lambda(a'x + b'y) = 0,$$
$$bx + cy + \lambda(b'x + c'y) = 0,$$
$$cx + dy + \lambda(c'x + d'y) = 0.$$

Solving these equations linearly for x, y, λx, λy, and then equating the product of x by λy to the product of y by λx, we get for the required condition

$$\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ c, & d, & c' \end{vmatrix} \times \begin{vmatrix} a', & b', & b \\ b', & c', & c \\ c', & d', & d \end{vmatrix} = \begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ c, & d, & d' \end{vmatrix} \times \begin{vmatrix} a', & b', & a \\ b', & c', & b \\ c', & d', & c \end{vmatrix};$$

or $$\{b(bc') + c(ca') + d(ab')\} \times \{a'(cd') + b'(db') + c'(bc')\}$$
$$= \{b'(bc') + c'(ca') + d'(ab')\} \times \{a(cd') + b(db') + c(bc')\},$$

whence $$-Q = (bc')^3 + (ca')^2(cd') + (bd')^2(ab') - 3(ab')(bc')(cd')$$
$$-(ad')(bc')^2 - (ad')(ab')(cd').$$

In a different form, by eliminating λ, we have the equations

$$\frac{ax + by}{a'x + b'y} = \frac{bx + cy}{b'x + c'y} = \frac{cx + dy}{c'x + d'y},$$

which are

$$(ab')x^2 + (ac')xy + (bc')y^2 = 0,$$
$$(ac')x^2 + [(ad') + (bc')]xy + (bd')y^2 = 0,$$
$$(bc')x^2 + (bd')xy + (cd')y^2 = 0;$$

therefore $$\begin{vmatrix} (ab'), & (ac'), & (bc') \\ (ac'), & (ad') + (bc'), & (bd') \\ (bc'), & (bd'), & (cd') \end{vmatrix} = Q.$$

Again, eliminating $x : y$ from the original three equations we get

$$\frac{a + \lambda a'}{b + \lambda b'} = \frac{b + \lambda b'}{c + \lambda c'} = \frac{c + \lambda c'}{d + \lambda d'},$$

whence $$ac - b^2 + \lambda(ac' + ca' - 2bb') + \lambda^2(a'c' - b'^2) = 0,$$
$$ad - bc + \lambda(ad' + da' - bc' - cb') + \lambda^2(a'd' - b'c') = 0,$$
$$bd - c^2 + \lambda(bd' + db' - 2cc') + \lambda^2(b'd' - c'^2) = 0,$$

so that $$\begin{vmatrix} ac-b^2, & ac'+ca'-2bb', & a'c'-b'^2 \\ ad-bc, & ad'+da'-bc'-cb', & a'd'-b'c' \\ bd-c^2, & bd'+db'-2cc' & b'd'-c'^2 \end{vmatrix} = Q;$$

or yet again, eliminating dialytically,

$$\begin{vmatrix} 0, & 0, & a, & b, & a', & b' \\ 0, & 0, & b, & c, & b', & c' \\ 0, & 0, & c, & d, & c', & d' \\ a, & b, & a', & b', & 0, & 0 \\ b, & c, & b', & c', & 0, & 0 \\ c, & d, & c', & d', & 0, & 0 \end{vmatrix} = Q.$$

If, as in the last article, we give a, b, &c. the values $A - C$, $-C$, &c., this Q would become $(BC')(CA')(AB')$. If then we subtract twenty-seven times this quantity from $\{(ad') - 3(bc')\}^3$, we get the resultant in the form

$$R = (ad')^3 - 9(ad')^2(bc') + 27(ca')^2(cd') + 27(db')^2(ab') - 81(ab')(bc')(cd') - 27(ad')(ab')(cd'),$$

a result which agrees with that of Art. 80, it being remembered that there the cubics were written without binomial coefficients.

215. We have, in Art. 213, formed the invariant P of the system $Ax^3 + By^3 + Cz^3$, $A'x^3 + B'y^3 + C'z^3$, by first reducing them to functions of two variables, and then calculating the value of $(ad') - 3(bc')$. We shall, for the sake of establishing a useful general principle, give another way of making the same calculation. We know that we may substitute in any binary quantic $\frac{d}{dy}$, $-\frac{d}{dx}$ for x and y, and so obtain an invariantive symbol of operation. Now when this change is made in a function expressed in terms of x, y, z, where z is $-(x+y)$, we must for z write $\frac{d}{dx} - \frac{d}{dy}$. And when the operation is performed on a function similarly expressed, since its differential with respect to x will be $\frac{d}{dx} + \frac{dz}{dx}\frac{d}{dz}$, or, in virtue of the relation between x, y, z, $\frac{d}{dx} - \frac{d}{dz}$, we see that the rule may be expressed, that in any

covariant we may substitute for x, y, z respectively

$$\frac{d}{dy} - \frac{d}{dz}, \quad \frac{d}{dz} - \frac{d}{dx}, \quad \frac{d}{dx} - \frac{d}{dy},$$

and so obtain an operative symbol which we may apply to any covariant expressed in terms of x, y, z, without first reducing it to a function of two variables. Thus, in the present case, we find the invariant P by operating on $A'x^3 + B'y^3 + C'z^3$, with

$$A\left(\frac{d}{dy} - \frac{d}{dz}\right)^3 + B\left(\frac{d}{dz} - \frac{d}{dx}\right)^3 + C\left(\frac{d}{dx} - \frac{d}{dy}\right)^3,$$

and the result only differs by a numerical factor from the foregoing expression $(BC') + (CA') + (AB')$.

In like manner we find that, in the symbolical notation, $\overline{12}$, as applied to a function expressed in terms of x, y, z, denotes

$$\begin{vmatrix} 1, & 1, & 1 \\ \frac{d}{dx_1}, & \frac{d}{dy_1}, & \frac{d}{dz_1} \\ \frac{d}{dx_2}, & \frac{d}{dy_2}, & \frac{d}{dz_2} \end{vmatrix}.$$

The Jacobian of the system of two cubics is a combinantive covariant, whose value is

$$\begin{vmatrix} 1, & 1, & 1 \\ Ax^2, & By^2, & Cz^2 \\ A'x^2, & B'y^2, & C'z^2 \end{vmatrix},$$

or $$(BC')\,y^2z^2 + (CA')\,z^2x^2 + (AB')\,x^2y^2.$$

This is a quartic, for which the two invariants may be expressed in terms of the combinants which have been enumerated already. Putting in for z^2, $(x+y)^2$, and multiplying the Jacobian by six to avoid fractions, we get

$$a = 6\,(CA'),\ b = 3\,(CA'),\ e = 6\,(BC'),\ d = 3\,(BC'),$$
$$c = (BC') + (CA') + (AB') = P,$$

whence $S = 3P^2$, $T = 54Q - P^3$. We have seen that the discriminant of a biquadratic is $S^3 - 27T^2$. The discriminant of the Jacobian, therefore, is proportional to $Q\,(P^3 - 27\,Q)$, which agrees with Art. 180.

216. The value of the Jacobian of the cubics

$$u = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$
$$v = a'x^3 + 3b'x^2y + 3c'xy^2 + d'y^3,$$

is $(ab')x^4 + 2(ac')x^3y + [(ad') + 3(bc')]x^2y^2 + 2(bd')xy^3 + (cd')y^4$; writing this

$$J = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4,$$

we have, since, $P = (ad') - 3(bc')$,

$$(ab') = a_0, \quad (ac') = 2a_1, \quad (ad') = 3a_2 + \tfrac{1}{2}P,$$
$$(bc') = a_2 - \tfrac{1}{6}P, \quad (bd') = 2a_3, \quad (cd') = a_4.$$

Accordingly, the identity

$$(bc')(ad') + (ca')(bd') + (ab')(cd') = 0$$

gives us $a_0a_4 - 4a_1a_3 + 3a_2^2 = \frac{1}{12}P^2$;

also substituting these values in the determinant form of Q expressed by means of them, p. 206, we find

$$a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_4a_1^2 - a_2^3 = \tfrac{1}{4}(Q - \tfrac{1}{64}P^3).$$

The covariant $\overline{12}^2$ of u and v is, in full,

$$(ac' + ca' - 2bb')x^2 + (ad' + da' - bc' - cb')xy + (bd' + db' - 2cc')y^2,$$

which we write $= H_1 = \alpha_1x^2 + 2\beta_1xy + \gamma_1y^2$,

thus $ac' + ca' - 2bb' = \alpha_1$, $\quad ad' + cb' - 2bc' = \beta_1 + \tfrac{1}{2}P$,

$$bc' + da' - 2cb' = \beta_1 - \tfrac{1}{2}P, \quad bd' + db' - 2cc' = \gamma_1.$$

If the Hessians of u and v be written respectively

$$H = \alpha x^2 + 2\beta xy + \gamma y^2, \quad H' = \alpha' x^2 + 2\beta' xy + \gamma' y^2,$$

where $ac - b^2 = \alpha$, &c.,

we find, as in Ex. 6, p. 24,

$$8(\alpha\gamma' + \gamma\alpha' - 2\beta\beta') = 4(\alpha_1\gamma_1 - \beta_1^2) + P^2.$$

The results of operating with either cubic on the Jacobian are two linear covariants, which may be compared with the results of operating on the cubics with the Hessians; it is easy to see that we have thus the different ways of writing them,

$$ac' - 2\beta b' + \gamma a' = -(a\gamma_1 - 2b\beta_1 + c\alpha_1) = \tfrac{2}{3}(aa_3 - 3ba_2 + 3ca_1 - da_0)$$
$$= -\{b(bc') + c(ca') + d(ab')\} = L_1,$$
$$ad' - 2\beta c' + \gamma b' = \text{&c.} = L_2,$$
$$c\alpha' - 2b\beta' + a\gamma' = \text{&c.} = L'_1, \quad d\alpha' - 2c\beta' + b\gamma' = \text{&c.} = L'_2,$$

denoting these two linear covariants by

$$L = L_1 x + L_2 y, \quad L' = L'_1 x + L'_2 y,$$

or in determinant form

$$L = \begin{vmatrix} a'x + b'y, & a, & b \\ b'x + c'y, & b, & c \\ c'x + d'y, & c, & d \end{vmatrix}, \quad L' = \begin{vmatrix} ax + by, & a', & b' \\ bx + cy, & b', & c' \\ cx + dy, & c', & d' \end{vmatrix}.$$

It is obvious that their determinant

$$\begin{vmatrix} L'_1, & L'_2 \\ L_1, & L_2 \end{vmatrix} = \begin{vmatrix} \alpha, & 2\beta, & \gamma \\ \alpha_1, & 2\beta_1, & \gamma_1 \\ \alpha', & 2\beta', & \gamma' \end{vmatrix} = Q.$$

Ex. For the forms in Art. 215, the above quadricovariants are

$$BCyz + CAzx + ABxy,$$

$$(BC' + CB')\, yz + (CA' + AC')\, zx + (AB' + BA')\, xy,$$

$$B'C'yz + C'A'zx + A'B'xy,$$

and the above linear covariants are found by operating by the method of that article to be

$$A'BCx + B'CAy + C'ABz \text{ and } AB'C'x + BC'A'y + CA'B'z.$$

217. There is another form in which the system of two cubics may be usefully discussed, viz.

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3, \quad bx^3 + 3cx^2y + 3dxy^2 + ey^3.$$

In other words, the cubics may be so transformed as to become the differential coefficients of the same quartic, with regard to x and y respectively. We can infer, from counting the constants, that the proposed form is sufficiently general; but the possibility of the transformation will be more clearly seen if we consider the two linear covariants just obtained: if we make $a', b', c', d' = b, c, d, e$, we have $L' = Tx$, $L = Ty$. Thus we see that, in order to effect the proposed transformation, we are to take these two linear covariants for the new variables.

If u and v be the differentials with regard to x and y of the same quartic, the quartic itself can only differ by a numerical factor from $xu + yv$; and in fact $L'u + Lv$ is immediately seen to be a combinant, as being a function of the determinants (ab') &c. The leading term in $L'u + Lv$ is

$$(ab')(bc') + (ab')(ad') - (ac')^2.$$

Whence, substituting the values given in last article this leading term is found to be $4(a_0a_2 - a_1^2) + \frac{1}{3}Pa_0$. But $a_0a_2 - a_1^2$ is the source of the Hessian of the Jacobian; accordingly

$$12H(J) + PJ = 3(L'u + Lv).$$

Thus, in order to obtain the quartic, having for its differentials the two given cubics, we must add twelve times the Hessian of the Jacobian to the Jacobian itself multiplied by P.

For the system of cubics considered in this article, J is the Hessian form of the quartic,

$$P = ae - 4bd + 3c^2,$$

and

$$Q = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}^2.$$

218. If we have any invariant of a single quantic, and if we perform on it the operation $a'\frac{d}{da} + b'\frac{d}{db} + \&c.$, we obtain an invariant of the system of two quantics of the same degree. For if we form the corresponding invariant of $u + \lambda v$, the coefficients of the different powers of λ will evidently be invariants of the system. Thus let the discriminant of $u + \lambda v$ be $D + 4\lambda M + 6\lambda^2 N + 4\lambda^3 M' + \lambda^4 D'$, and we have the three new invariants M, N, M', whose orders in the coefficients are (3, 1), (2, 2), (1, 3).

If in general we form any invariant of $u + \lambda v$, and then form any invariant of this again considered as a function of λ, the result will be a combinant of the system u, v; that is to say, it will not be altered if we substitute $lu + mv$, $l'u + m'v$ for u, v. For, by this substitution, we get the corresponding invariant of $(l + \lambda l')u + (m + \lambda m')v$, which is equivalent to a linear transformation of λ, by which the invariants of the function of λ will not be altered. If then it be required to calculate the invariants of the biquadratic, which we have found for the discriminant in the case of two cubics, we may, without loss of generality, take instead of u and v two quantics of the system $u + \lambda v$ which have square factors, taking x and y for these factors; and so write $u = ax^3 + 3bx^2y$, $v = 3cxy^2 + dy^3$. For this system we have $P = ad - 3bc$, $Q = b^2c^2(ad - bc)$; the

resultant $P^3 - 27Q$ being $a^2d^2(ad - 9bc)$. Now, for this form, the biquadratic is $4ac^3\lambda^3 + (a^2d^2 - 6abcd - 3b^2c^2)\lambda^2 + 4db^3\lambda$; or multiplying by six to avoid fractions $D = D' = 0$, $M = 6db^3$, $M' = 6ac^3$, $N = a^2d^2 - 6abcd - 3b^2c^2 = P^2 - 12b^2c^2$. Hence,

$$S = 3N^2 - 4MM' = 3P(P^3 - 24Q);$$

$$T = 2NMM' - N^3 = -(P^6 - 36P^3Q + 216Q^2),$$

whence the discriminant of the biquadratic $S^3 - 27T^2$ is proportional to $Q^3(P^3 - 27Q)$, which agrees with Art. 183.

The method used above is evidently also applicable to covariants. Thus let the Hessian of $u + \lambda v$ be $H + \lambda H_1 + \lambda^2 H'$, and we are led to the intermediate quadratic covariant H_1, whose leader is $ac' + ca' - 2bb'$. The covariants and invariants of the system of three quadratics, just mentioned, are also covariants and invariants of the system of two cubics. Thus if we take the Jacobian of each pair of quadratics, we have three quadric covariants of the orders (3, 1), (2, 2), (1, 3). We have seen (Art. 167) that a cubic and its Hessian have the same discriminant, and therefore we may identify the discriminant of $H + \lambda H_1 + \lambda^2 H'$ with the expression already found for the discriminant of $u + \lambda v$. Now if u, v, w be three quadratics, the discriminant of $\lambda u + \mu v + \nu w$ is plainly $\lambda^2 D_{11} + 2\lambda\mu D_{12} +$ &c. Thus for the system under consideration, we see that D_{12}, D_{23}, can only differ by a factor from M, M' already enumerated; and we have the two invariants D_{22}, $2D_{13}$ whose sum similarly is, to a factor, the same with N. Another relation was already found (Art. 216) between them and P. These invariants, therefore are not new but can be expressed each in terms of N and the combinant P. The expression is most easily arrived at by taking the particular case already considered $u = ax^3 + 3bx^2y$, $v = 3cxy^2 + dy^3$, in which case we have

$$H_1 = acx^2 + (ad - bc)xy + bdy^2,\ 2D_{13} = b^2c^2,\ 4D_{22} = 6abcd - a^2d^2 - b^2c^2.$$

Thus we find, for the case when the discriminants of both cubics vanish, the relations $P^2 - N = 24D_{13}$, $P^2 + 2N = -12D_{22}$; and it can easily be verified that these relations are true in general.

Lastly, we may form the invariant R of the system of three quadratics, but we have found this already to be the combinant Q.

219. We are able now to give a list of the covariants of the system, which the investigations of Clebsch and Gordan show to be complete.* There is *one quartic covariant* (1, 1), the Jacobian (Art. 216). There are *six cubic covariants*, the two cubics themselves (1, 0) (0, 1), their two cubicovariants (3, 0) (0, 3), and the two Jacobians (2, 1) (1, 2) of either cubic combined with the Hessian of the other. For the canonical form, the last four covariants are included in the form

$$\begin{vmatrix} 1 & , & 1 & , & 1 \\ Ax^2 & , & By^2 & , & Cz^2 \\ A(By+Cz), & & B(Cz+Ax), & & C(Ax+By) \end{vmatrix},$$

according as we accentuate the coefficients in neither, either, or both of the last two rows. There are *six quadratic covariants*, viz. the two Hessians (2, 0), (0, 2), and the intermediate covariant (1, 1), these three being for the canonical form $\Sigma BCyz$, $\Sigma B'C'yz$, $\Sigma(BC'+B'C)yz$; and for the remaining three covariants (3, 1), (2, 2), (1, 3), we may take either the Jacobians of each pair of these (Art. 218), or the results obtained by operating with each on the quartic covariant.

There are *six linear covariants*, viz. the two (1, 2), (2, 1) considered Art. 216; two (3, 2), (2, 3) obtained by operating with the Hessian of either cubic on the cubicovariant of the other; two (1, 4), (4, 1) obtained by operating with either cubic on the square of the Hessian of the other. Lastly, there are *seven invariants*, viz. the two discriminants (4, 0), (0, 4), the combinants P, Q, (1, 1), (3, 3) (Art. 213), and the invariants M, M', N, of Art. 218. Of the preceding invariants P and Q are skew. We have in Art. 218 connected P^2 with the invariants D_{13}, D_{22} of that article, and the expressions there given for the S and T of the biquadratic are, in fact, expressions for PQ and Q^2 in terms of N, M', &c. We can also connect Q^2 with the functions D_{11}, D_{12}, &c. if we remember that, as was remarked (Art. 218), Q is the invariant $\overline{12}.\overline{23}.\overline{31}$ of a

* Of the eight linear covariants which they supposed to be irreducible, Sylvester has shown that two are not, and d'Ovidio and Gerbaldi have developed their expressions, *Atti di Torino*, XV. 267.

system of three quadratics, and that it was proved at the end of Art. 193 that double the square of this invariant is

$$\begin{vmatrix} D_{11}, & D_{12}, & D_{13} \\ D_{12}, & D_{22}, & D_{23} \\ D_{13}, & D_{23}, & D_{33} \end{vmatrix}.$$

219*a*. Geometrically, the two cubics may be taken as two triads of points on a conic, as in the case of Art. 198. When any point $O(x'y')$ on the conic is joined to the points of the triad (u), the joining lines meet the sides of the triangle (u) in points forming a triangle in projection with (u), and it is easily seen that the first emanant with regard to x', y' of the cubic u, whose roots are represented by the points of the triad, is the axis of the projection. Similarly, O may be made the centre of a projection for v, and the axis corresponding be found; these two axes intersect in a point O', and as O moves round its conic, O' describes another conic which intersects the conic O in four points representing the roots of the Jacobian of u and v. The four corresponding points O can without difficulty be determined.

The roots of the covariant H_1 are represented by two points on the original conic whose axes with respect to the two triads are conjugate lines with respect to that conic.

In Art. 198 it was seen that for a system of a cubic u and a quadric V the simplest linear covariant is the centre of the projection with regard to u whose axis passes through the pole of V. If we calculate for the triad consisting of this centre and the two points where V meets the conic, the combinant P with the triad u, we find $P=0$. It follows from symmetry that each of the vertices of the former triangle is the centre of a projection with regard to u, of which the axis is conjugate to the opposite side; and that such relation holds mutually between u and v when $P=0$. Another way of stating the condition $P=0$ is that the triads are such that H and its pole are axis and centre of a projection with regard to v, and H' and its pole, axis and centre of a projection with regard to u.

The simplest linear covariants discussed above are represented as follows: L is the centre of the projection with respect to v,

whose axis is the line joining the poles of H, H'; and L' the centre of the projection with respect to u of the same line. The identities established, Art. 216, show that L is also centre of the projection with respect to u, whose axis joins the poles of H, H_1, and L' centre of the projection with respect to v, whose axis joins the poles of H_1, H'. Thus the pole of H_1 can be constructed.

If the points L, L' coincide, H, H_1, H' are concurrent lines, and the line joining their poles is the axis of projection for the L point with respect to both triads and the combinant Q vanishes.

219*b*. *System of four cubics.* It is desirable to generalise to this case the theorems given p. 181 for three quadrics. For the determinant of the system, if we denote the combinant $(a_1d_2)-3(b_1c_2)$ by P_{12}, &c., we have, by Art. 33, Ex. 4, the relation

$$P_{23}P_{14}+P_{31}P_{24}+P_{12}P_{34}=-3(a_1b_2c_3d_4),$$

and this becomes when a_4, &c., are replaced by y^3, &c.

$$P_{23}u_1+P_{31}u_2+P_{12}u_3$$

$$=3\begin{vmatrix} a_1, & a_2, & a_3, & y^3 \\ b_1, & b_2, & b_3, & -xy^2 \\ c_1, & c_2, & c_3, & x^2y \\ d_1, & d_2, & d_3, & -x^3 \end{vmatrix} =-3\begin{vmatrix} a_1x+b_1y, & a_2x+b_2y, & a_3x+b_3y \\ b_1x+c_1y, & b_2x+c_2y, & b_3x+c_3y \\ c_1x+d_1y, & c_2x+d_2y, & c_3x+d_3y \end{vmatrix}.$$

This is the cubic which determines the values of x', y', so that $\lambda_1u_1+\lambda_2u_2+\lambda_3u_3=(xy'-x'y)^3$ be a perfect cube, see *Curves* (Art. 216*a* &c.), and the discussion of unicursal cubics there given furnishes an additional geometrical illustration of this theory. For four cubics, denoting the function just written by J_{123}, we have the equations

$$\begin{aligned} P_{34}u_2+P_{42}u_3+P_{23}u_4&=-3J_{234}, \\ P_{34}u_1\qquad\quad+P_{41}u_3+P_{13}u_4&=-3J_{341}, \\ P_{24}u_1+P_{41}u_2\qquad\quad+P_{12}u_4&=-3J_{412}, \\ P_{23}u_1+P_{31}u_2+P_{12}u_3\qquad\quad&=-3J_{123}, \end{aligned}$$

a linear transformation in four variables.

If J_{12} signify the Jacobian of the cubics a_1x^3+ &c., a_2x^3+ &c., we have six such Jacobians each a quartic in x, y, which we may write

$$J_{12}=a_{12}x^4+4b_{12}x^3y+6c_{12}x^2y^2+4d_{12}xy^3+e_{12}y^4, \text{ &c.},$$
$$a_{12}=(a_1b_2),\quad 2b_{12}=(a_1c_2), \text{ &c.}$$

Now it is easily found, as in Exs. 3, 4, Art. 33, that

$$\begin{aligned}(a_1b_2c_3d_4)= &\ a_{12}e_{34}+a_{13}e_{42}+a_{14}e_{23}+a_{34}e_{12}+a_{42}e_{13}+a_{23}e_{14}\\ =&-4\{b_{12}d_{34}+b_{13}d_{42}+b_{14}d_{23}+b_{34}d_{12}+b_{42}d_{13}+b_{23}d_{14}\}\\ =&\ 12\,(c_{12}c_{34}+c_{13}c_{42}+c_{14}c_{22}).\end{aligned}$$

Hence, and from the vanishing combinations,

$$a_{12}b_{34}+a_{34}b_{12}+\text{ &c.}=0, \text{ &c.},$$

the linear relation had by dialytic elimination of x^4, x^3y, &c., between the six functions J_{12} is found to be

$$P_{12}J_{34}+P_{13}J_{42}+P_{14}J_{23}+P_{34}J_{12}+P_{42}J_{13}+P_{23}J_{14}=0.$$

Other relations can be established connecting the functions H with these, as at p. 181, on which we need not delay. For instance, we might show for five cubics, with an obvious notation, that

$$2J_{12}J_{345}=\begin{vmatrix} u_3, & H_{13}, & H_{23}\\ u_4, & H_{14}, & H_{24}\\ u_5, & H_{15}, & H_{25}\end{vmatrix}.$$

Also for four that $J_{23}J_{14}+J_{31}J_{24}+J_{12}J_{34}=0$.*

219*c*. The system of four cubics furnishes another geometrical view of binary cubic systems. For, we can by linear combinations alter the functions to x^3, x^2y, xy^2, y^3 and introduce these as new variables x, y, z, w, the coordinates of a point in space which depends on a single parameter t. The relations between them, and t, may be written $x:y=y:z=z:w=t$; thus the point t is restricted to a twisted cubic, and the roots of any binary cubic are the parameters of the points of intersection of a plane with the twisted cubic (compare *Surfaces*, Art. 337 *et seq*).

For any point t on the curve the osculating plane is

$$x-3yt+3zt^2-wt^3=0,$$

so that the points whose osculating planes meet in any point

* Dr. F. Lindemann's paper (*Math. Annalen*, vol. XXIII. p. 111) on the geometric exposition of binary forms includes the formulæ of this article.

$x'y'z'w'$ are determined by $x' - 3y't + 3z't^2 - w't^3 = 0$, or the points of contact lie in the plane $w'x - 3z'y + 3y'z - x'w = 0$, but this plane passes through $x'y'z'w'$.

Hence, in any plane F, $ax + 3by + 3cz + dw = 0$ there is a point O whose coordinates are $d, -c, b, -a$, which is the intersection of osculating planes at the points in which it meets the curve, and the point and plane may be said to correspond. When O lies in the plane F' corresponding to a point O', the above relation shows that O' lies in the plane F corresponding to O; and the line OO' possesses an invariant relation to the curve. For two such planes F, F' the P invariant vanishes.

Two equations of the line meeting the curve in the points t_1, t_2 are $x - y(t_1 + t_2) + zt_1t_2 = 0$, $y - z(t_1 + t_2) + wt_1t_2 = 0$, if we express that this line passes through the point O, the parameters t_1, t_2 of the points in which the chord of the curve through O meets the curve are the roots of the quadratic

$$(ac - b^2)\, t^2 + (ad - bc)\, t + bd - c^2 = 0,$$

thus the Hessian of the binary cubic is represented by the intersections of the chord through O with the curve.

The determinant, whose constituents are the coordinates of the collinear points O, t_1, t_2, and of any other point, vanishes identically. Hence, by the first formula of last Article, if we put

$$ax_1 + 3by_1 + 3cz_1 + dw_1 = \lambda_1,$$

$$ax_2 + 3by_2 + 3cz_2 + dw_2 = \lambda_2,$$

we have identically

$$\lambda_1 (xw_2 - 3yz_2 + 3zy_2 - wx_2) - \lambda_2 (xw_1 - 3yz_1 + 3zy_1 - wx_1)$$

$$= (x_1w_2 - 3y_1z_2 + 3z_1y_2 - w_1x_2)(ax + 3by + 3cz + dw).$$

Hence the plane F contains the line of intersection of the osculating planes at the Hessian points.

The plane through the intersection of the osculating planes

$$\lambda_1 (xw_2 - 3yz_2 + 3zy_2 - wx_2) + \lambda_2 (xw_1 - 3yz_1 + 3zy_1 - wx_1) = 0$$

which is harmonic to the former is easily found to represent the cubicovariant, and having the same Hessian, its corresponding

point is the harmonic conjugate of O with regard to the Hessian points.

219*d*. A second plane Φ gives rise to a similar system, and besides to the results of combining the two. The coordinates of the tangent line to the curve at any point are t^2, $-2t^3$, t^4, $3t^2$, $2t$, 1, whence we find that t is a root of the Jacobian of F, Φ for any point on the curve whose tangent meets the line in which the planes intersect. The line joining OO' meets the tangents at the same points.

The parameters of the points in which a chord through any point on OO' meets the curve are given by

$$\lambda^2 H + 2\lambda\mu H_1 + \mu^2 H' = 0.$$

From any point given on the curve, two chords can be drawn to meet OO', and if $H_1 = 0$, these chords meet OO' in points harmonically conjugate to O and O'.

The simplest linear covariants are represented by the point where the plane through O containing the chord through O' meets the curve again, and that in which the plane through O' containing the chord through O meets the curve again.

The invariant P vanishes for two planes if the point corresponding to one lie in the other, in which case the line $F\Phi$ becomes identical with its corresponding line OO'.

The combinant Q vanishes when it is possible to draw an osculating plane through the line $F\Phi$, in this case it is easy to see the corresponding line OO' meets the curve and the two points which represent the simplest linear covariants coincide.*

219*e*. *System of a quartic and cubic.* This system consists of sixty-one forms†: 1 sextic, 2 quintics, 5 quartics, 8 cubics, 10 quadrics, 15 linear, and 20 invariant functions. An invariant of the third degree in the coefficients of the quartic, and of the second in those of the cubic, can at once be written

* See a paper on this geometrical representation by Mr. W. R. W. Roberts, *Proc. Lond. Math. Society*, vol. XIII. Also Dr. Lindemann's paper, Note p. 216.

† In Dr. Gundelfinger's inaugural dissertation the number was assigned as 64, but Dr. Sylvester has since shown (*Comptes Rendus*, t. LXXXVII.) that two of the quadratic and one of the linear covariants are reducible.

down from the determinant form of the sextic covariant, p. 192, being the result of operating with that covariant on the square of the cubic,

$$\begin{vmatrix} a', & -3b', & 3c', & -d', & 0 \\ 0, & a', & -3b', & 3c', & -d' \\ 0, & a, & -3b, & 3c, & -d \\ a, & -2b, & 0, & 2d, & -e \\ b, & -3c & 3d, & -e, & 0 \end{vmatrix}.$$

The same determinant with only one row so altered is a covariant cubic, and another of the same degree in the quartic coefficients may be got from the skew invariant, Art. 219.

The two simplest linear covariants are

$$a'(dx+ey)-3b'(cx+dy)+3c'(bx+cy)-d'(ax+by),$$

and the corresponding one for the Hessian of the quartic. The determinant just written is the eliminant of these two.

Operating with the quartic on the square of the Hessian of the cubic, gives an invariant of the first degree in the coefficients, of the former.

220. *System of two quartics.* We consider here chiefly the invariants of this system, which are also combinants. We have seen, Art. 218, that the invariants of any invariant of $\lambda u+\mu v$ are combinants. Let us form the S and T of $\lambda u+\mu v$ and write them

$$S\lambda^2+\Sigma\lambda\mu+S'\mu^2,\ T\lambda^3+t\lambda^2\mu+t'\lambda\mu^2+T'\mu^3;$$

then if we form (Art. 198) the invariants and covariants of this cubic and quadratic, we get combinants of the system of two quartics.

Thus the discriminant of the quadratic is $\Sigma^2-4SS'=$

$$\begin{vmatrix} a, & 4b, & 6c, & 4d, & e \\ a', & 4b', & 6c', & 4d', & e' \end{vmatrix} \begin{vmatrix} e', & -d', & c', & -b', & a' \\ e, & -d, & c, & -b, & a \end{vmatrix}$$

$$=(ae')^2+16(bd')^2+12(ac')(ce')-48(bc')(cd')-8(ab')(de')-8(ad')(be'),$$

which we shall refer to as the combinant A.

It will be convenient to use the abbreviations $(ab')=\alpha$, $(de')=\alpha'$, $(ad')=\beta$, $(be')=\beta'$, $(ac')=\lambda$, $(ce')=\lambda'$, $(bc')=\mu$, $(cd')=\mu'$, $(ae')=\gamma$, $(bd')=\delta$. We have then

$$A=12\lambda\lambda'-48\mu\mu'+\gamma^2+16\delta^2-8\alpha\alpha'-8\beta\beta'.$$

We can find by a different process an independent combinant of the same order in the coefficients. The Jacobian of the two

quartics, J, is

$$\alpha x^6+3\lambda x^5y+(3\beta+6\mu)x^4y^2+(\gamma+8\delta)x^3y^3+(3\beta'+6\mu')x^2y^4+3\lambda' xy^5+\alpha' y^6.$$

Again a combinant quadratic P or $\overline{12}^3$ is

$$(\beta-3\mu)\,x^2+(\gamma-2\delta)\,xy+(\beta'-3\mu')\,y^2.$$

The sextic J has an invariant of the second order in its coefficients (Art. 141), and the discriminant of P is of like order, neither being identical with A.

The condition that u and v be derived functions of a quintic, is the vanishing of $B=\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ a', & b', & c', & d' \\ b', & c', & d', & e' \end{vmatrix}$: hence, or as in Art. 219b, we find $B=\lambda\lambda'-\mu\mu'-\beta\mu'-\beta'\mu+\delta^2-\alpha\alpha'$; the invariant of J will be found to be $A+48B$, and the discriminant of P to be $A-12B$.

The resultant of u, v, found by expanding the determinant of Art. 84 is

$$\begin{aligned} R=\ & 1296\lambda^2\lambda'^2-3456\,(\alpha\mu\lambda'^2+\alpha'\mu'\lambda^2)-1152\,(\alpha\beta\lambda'^2+\alpha'\beta'\lambda^2) \\ & -72\gamma^2\lambda\lambda'-576\gamma\delta\lambda\lambda'+9216\alpha\alpha'\mu\mu'+96\gamma\,(\beta^2\lambda'+\beta'^2\lambda) \\ & +288\gamma\,(\alpha\beta'\lambda'+\alpha'\beta\lambda)+1536\delta(\alpha\beta'\mu'+\alpha'\beta\mu)+3072\alpha\alpha'\,(\beta\mu'+\beta'\mu) \\ & +\gamma^4-48\alpha\alpha'\gamma^2-16\beta\beta'\gamma^2-256\alpha\alpha'\gamma\delta+512\alpha^2\alpha'^2 \\ & -256\,(\alpha\beta'^3+\alpha'\beta^3)-4096\alpha\alpha'\delta^2. \end{aligned}$$

In terms of the preceding combinants can be expressed the combinant which we have called T (Art. 182), but which, in order to avoid confusion, we shall now call C; and which expresses the condition that a quartic of the system $u+\lambda v$ can have two square factors. Such a combinant must vanish if v reduce to the single term $c'x^2y^2$. In such a case α, α', β, β', γ, δ all vanish; and we have $A=12\lambda\lambda'-48\mu\mu'$, $B=\lambda\lambda'-\mu\mu'$, $R=1296\lambda^2\lambda'^2$; hence we see that $(A-48B)^2-R$ is a combinant which vanishes on this supposition. And since it is of the same order that we have seen, Art. 182, that T must be, it is identical with it. Using the values already given for A, B, R, we find that $(A-48B)^2-R=128\,C$, where

$$\begin{aligned} C=\ & -27\gamma(\lambda\mu'^2+\lambda'\mu^2)+18(\beta^2\mu'^2+\beta'^2\mu^2)+18\delta^2\lambda\lambda'+36\gamma\delta\mu\mu'-36\alpha\alpha'\mu\mu' \\ & +18\gamma\delta\lambda\lambda'-9\gamma^2\mu\mu'-3\gamma\,(\alpha\beta'\lambda'+\alpha'\beta\lambda)-24\delta^2\,(\beta\mu'+\beta'\mu) \\ & -6\delta\,(\beta^2\lambda'+\beta'^2\lambda)-6\delta\,(\alpha\beta'\lambda'+\alpha'\beta\lambda)+2\,(\alpha\beta'^3+\alpha'\beta^3)+\alpha\alpha'\gamma^2 \\ & +4\alpha^2\alpha'^2-2\alpha\alpha'\gamma\delta+4\gamma\delta^3+16\alpha\alpha'\delta^2+8\delta^4. \end{aligned}$$

Again, if we form the invariant which we called I (Art. 198) of the quadratic and cubic at the beginning of this article, it will be found that

$$(A+48B)(A-16B)-R=-128I,$$

a formula obtained by Mr. Warren, *Quarterly Journal*, Vol. VII., p. 70.

221. We consider next D, the resultant of the cubic and quadratic, and E the discriminant of the cubic. D is the invariant which we have called S (Art. 182). It may be mentioned, that besides the methods already indicated for calculating that invariant in general, the following may be used. It is required to find the condition that λ can be determined so that the three expressions $u_{11}+\lambda v_{11}$, $u_{12}+\lambda v_{12}$, $u_{22}+\lambda v_{22}$ can be made to vanish together. Now we may multiply each of these by the $2(n-2)$ terms x^{2n-5}, &c., of a quantic of the degree $2n-5$, and so obtain $6(n-2)$ equations, from which we can eliminate dialytically the $6(n-2)$ quantities x^{3n-7}, &c., λx^{3n-7},&c., and so obtain S in the form of a determinant. In the case of the system of two quartics,

$$\begin{aligned}
D=&-16\lambda^3\lambda'^3+48\lambda^2\lambda'^2\mu\mu'+6\lambda\lambda'\mu^2\mu'^2+16\mu^3\mu'^3+27\lambda^2\mu'^4+27\lambda'^2\mu^4\\
&+36\alpha\lambda\mu\lambda'^3+36\alpha'\lambda'\mu'\lambda^3+12\lambda^2\lambda'^2(\beta\mu'+\beta'\mu)-96\alpha\mu^2\lambda'^2\mu'\\
&-96\alpha'\mu'^2\lambda^2\mu-6\gamma\lambda\mu\mu'^3-6\gamma\mu^3\lambda'\mu'-36\delta\lambda^2\lambda'\mu'^2-36\delta\lambda\mu^2\lambda'^2\\
&-48\alpha\mu\mu'^4-48\alpha'\mu^4\mu'-24\delta\lambda\mu\mu'^3-24\delta\lambda'\mu'\mu^3+24\alpha\beta\lambda\lambda'^3\\
&+24\alpha'\beta'\lambda^3\lambda'-18\alpha\beta'\lambda'^2\mu^2-18\beta\alpha'\lambda^2\mu'^2-6\delta^2\lambda^2\lambda'^2+87\gamma^2\mu^2\mu'^2\\
&-\lambda\lambda'\mu\mu'(162\alpha\alpha'+90\beta\beta')-36\alpha\beta\mu'^4-36\alpha'\beta'\mu^4+96\delta^2\lambda\lambda'\mu\mu'\\
&+(228\alpha\alpha'-60\beta\beta')\mu^2\mu'^2-4\alpha^2\gamma\lambda'^3-4\alpha'^2\gamma\lambda^3-16\alpha\beta^2\lambda'^2\mu'\\
&-16\alpha'\beta'^2\lambda^2\mu-30\alpha\alpha'^2\lambda^2\mu'-30\alpha^2\alpha'\lambda'^2\mu-50\alpha\alpha'\lambda\lambda'(\beta\mu'+\beta'\mu)\\
&-20\delta\lambda\lambda'(\lambda\beta\alpha'+\lambda'\alpha\beta')+2\gamma\mu\mu'(\lambda\beta\alpha'+\lambda'\alpha\beta')+48\delta^2\alpha\mu\lambda'^2\\
&+48\delta^2\alpha'\mu'\lambda^2+24\alpha\beta'^2\mu\mu'^2+24\alpha'\beta^2\mu^2\mu'+56\alpha\beta\beta'\mu'^3+56\alpha'\beta\beta'\mu^3\\
&+240\alpha\alpha'\beta\mu\mu'^2+240\alpha\alpha'\mu^2\mu'\beta'+32\alpha\delta^2\mu'^3+32\alpha'\delta^2\mu^3+3\lambda^2\beta^2\alpha'^2\\
&+3\lambda'^2\alpha^2\beta'^2+24\alpha\alpha'(\beta^2\mu'^2+\beta'^2\mu^2)-6\alpha\alpha'\gamma^2\mu\mu'-12\alpha^2\alpha'^2\lambda\lambda'\\
&-30\alpha\alpha'\beta\beta'\lambda\lambda'+60\alpha\alpha'\lambda\lambda'\delta^2+12\beta\beta'\lambda\lambda'\delta^2\\
&+\{84\alpha^2\alpha'^2+120\alpha\alpha'\beta\beta'-12\beta^2\beta'^2\}\mu\mu'-192\alpha\alpha'\delta^2\mu\mu'+48\gamma\delta^2\mu\mu'\\
&-48\delta^4\lambda\lambda'+6\gamma\alpha\alpha'^2\beta\lambda+6\gamma\alpha^2\alpha'\beta'\lambda'+48\alpha^2\alpha'^2(\beta\mu'+\beta'\mu)\\
&+24\alpha\alpha'\delta(\beta^2\mu'+\beta'^2\mu)-96\alpha\alpha'\delta^2(\beta\mu'+\beta'\mu)-\alpha^2\alpha'^2\gamma^2-4\alpha\alpha'^2\beta^3\\
&-4\alpha^2\alpha'\beta'^3+8\alpha^3\alpha'^2-4\gamma\delta\alpha^2\alpha'^2-48\alpha^2\alpha'^2\delta^2-16\delta^2\alpha\alpha'\beta\beta'+64\alpha\alpha'\delta^4.
\end{aligned}$$

222. In studying the relations of these combinants, we may, without loss of generality, suppose one quartic to want the first two terms, and the other the last two; that is, we may write

$$u = ax^4 + 4bx^3y + 6cx^2y^2, \quad v = 6c'x^2y^2 + 4dxy^3 + ey^4.$$

To save room, we write $ae = l$, $bd = m$, $cc' = n$, $cad^2 + c'eb^2 = p^2$. We find then

$$A = (l - 4m)^2 + 12n(l - 4m),$$

$$B = m^2 - p^2 + n(l - m),$$

$$R = l^3(l - 16m) + 96l^2p^2 - 72nl^2(l + 8m) + 1296l^2n^2,$$

$$I = -l^2m^2 + 4lm^3 + p^2(l^2 - 2lm - 8m^2) + 6p^4 - n(l^3 + 6lm^2 - 16m^3) - 9nlp^2 + 12n^2(l^2 + lm - 2m^2),$$

$$D = -n^2\{9lm^2(l - 4m) - 6p^2(2l^2 - 6lm - 4m^2) - 27p^4 + 16n(l - m)^3\},$$

$$E = -l^2m^4 + 2lm^2p^2(l + 2m) - (l + 2m)^2p^4 - 2nlm^2(l + 2m)^2 + 4p^6 + 2np^2(l + 2m)^3 - 18np^2lm^2 - n^2(l + 2m)^4 + 36n^2lm^2(l + 2m) - 6np^4(l + 2m) - 6n^2p^2(l + 2m)^2 + 27n^2p^4 + 4n^3(l + 2m)^3 - 108n^3lm^2.$$

By the help of these values we can verify the equation

$$16B^3 - AB^2 - 2IB + E = D,$$

which expresses E in terms of invariants already found.

The Jacobian, with this form, wants the extreme terms. There is no difficulty, therefore, in calculating its discriminant, and thus verifying the theorem of Art. 180.

Finally, we have seen that a cubic and quadratic have a skew invariant M. The equation of connection given at the end of Art. 198, when applied to the case considered in this article, shows that the system of two quartics has a skew invariant M of the 9th order in the coefficients of each, whose square is given by the formula

$$M^2 = A(AE - I^2)^2 - 8D(I^3 - 9AIE + 54DE).$$

Mr. Burnside's method, reducing the theory of two quartics to that of three conics, discussed *Conics*, Art. 388, would have led us to the same results.

223. I have also sometimes found it convenient to suppose each quartic to be the sum of two fourth powers, so that for each the invariant T vanishes. Let the quartics be au^4+bv^4, $a'w^4+b'z^4$, where u is $\alpha_1 x+\beta_1 y$, &c. We use (12) to denote $\alpha_1\beta_2-\alpha_2\beta_1$, and employ the abbreviations

$$(12)(34)=L,\quad (13)(42)=M,\quad (14)(23)=N;$$

where it will be observed that we have identically $L+M+N=0$. Now the invariant S is got by substituting $\frac{d}{dy}$, $-\frac{d}{dx}$, for x, y in the quartic and operating on it with the result. If we operate in this way with u upon u the result vanishes; but if we operate on v the result is (12). We find then at once that the S of $\lambda U+\mu V$ is

$$\lambda^2 ab\,(12)^4+\lambda\mu\{aa'(13)^4+ab'(14)^4+ba'(23)^4+bb'(24)^4\}+\mu^2 a'b'(34)^4.$$

The combinant then which we have called A is

$$\{aa'(13)^4+ab'(14)^4+ba'(23)^4+bb'(24)^4\}^2-4aba'b'L^4.$$

In the same case B is found to be $-aba'b'L^2MN$.

The invariant T is found by operating on a quartic with its Hessian. But here the Hessian of U is $ab\,(12)^2u^2v^2$. We find then that the T of $\lambda U+\mu V$ is

$$\lambda^3\mu\{aba'(12)^2(13)^2(23)^2+abb'(12)^2(14)^2(24)^2\}$$
$$+\lambda\mu^2\{a'b'a(13)^2(14)^2(34)^2+a'b'b(23)^2(24)^2(34)^2\}.$$

Hence, we have immediately

$$E=-a^2b^2a'^2b'^2L^4\{aa'N^2(13)^4+ab'M^2(14)^4+ba'M^2(23)^4+bb'N^2(24)^4\}^2,$$

$$\begin{aligned}D=-a^2b^2a'^2b'^2L^6\{&a^2a'^2N^2(13)^8+a^2b'^2M^2(14)^8+b^2a'^2M^2(23)^8\\&+b^2b'^2N^2(24)^8-2MNa^2a'b'(13)^4(14)^4-2MNb^2a'b'(23)^4(24)^4\\&-2MNa'^2ab(13)^4(23)^4-2MNb'^2ab(14)^4(24)^4\\&+2M^2N^2aba'b'(M^2+N^2-2L^2)\},\end{aligned}$$

$$\begin{aligned}I=-aba'b'L^2[&a^2a'^2N^2(13)^8+a^2b'^2M^2(14)^8+b^2a'^2M^2(23)^8+b^2b'^2N^2(24)^8\\&+(M^2+N^2-2L^2)\{a^2a'b'(13)^4(14)^4+b^2a'b'(23)^4(24)^4\\&+a'^2ab(13)^4(23)^4+b'^2ab(14)^4(24)^4\}\\&+2M^2N^2(M^2+N^2-4L^2)aba'b'],\end{aligned}$$

by the help of which values we can verify the equation already obtained.

Sylvester has reduced by two the total number of independent forms for the system of two quartics which Gordan enumerated as thirty, viz. there are for each quartic the five forms u, H, J, S, T; twelve more forms are got by taking the operations $\overline{12}$, $\overline{12}^2$, $\overline{12}^3$, $\overline{12}^4$ performed on the pairs u, u'; u, H'; u', H; two more by $\overline{12}^3$, $\overline{12}^4$ performed on H, H'; and lastly, four more are obtained by operating on the sextic covariant of either quartic with the other quartic or with its Hessian. This gives in all eight invariants, eight quadratic, seven quartic, and five sextic covariants.

The following are examples added in further application of these principles to binary quantics of the first four degrees. In dealing with the sextic we shall return to the system of two quartics.

Ex. 1. If three quadratics be the three second derived functions of a single biquadratic function, express the relation between their coefficients.

With the notation of p. 181, $u=a_1x^2+2b_1xy+c_1y^2$, $v=a_2x^2+$&c., $2D_{rs}=a_rc_s+a_sc_r-2b_rb_s$, we must have

$$\begin{vmatrix} a_1, & b_1, & a_2, & b_2 \\ b_1, & c_1, & b_2, & c_2 \\ a_2, & b_2, & a_3, & b_3 \\ b_2, & c_2, & b_3, & c_3 \end{vmatrix} = 0.$$

This may be found by comparing coefficients when we effect a linear transformation on the quadratics. It may be found more rapidly in another form, as follows:

The general conic identity between the quadrics (p. 181) for

$$u = ax^2 + 2bxy + cy^2, \quad v = bx^2 + 2exy + dy^2, \quad w = cx^2 + 2dxy + ey^2,$$

has $D_{11} = ac - b^2$, $D_{22} = bd - c^2$, $D_{33} = ce - d^2$,

$$2D_{23} = be - cd, \quad 2D_{31} = ae + c^2 - 2bd, \quad 2D_{12} = ad - bc,$$

whence, by the identity $(ac-b^2)(ce-d^2)+(ad-bc)(cd-be)+(ae-bd)(bd-c^2)=0$, derived from the matrix $\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \end{vmatrix}$, we get

$$D_{11}D_{33} - 4D_{12}D_{23} + D_{22}(D_{22} + 2D_{31}) = 0.$$

Ex. 2. To give a geometrical signification of the preceding relation?

The equation $(ax + by + cz)\,x' + (bx + cy + dz)\,y' + (cx + dy + ez)\,z' = 0$ is that of the polar of (x', y', z') P to the conic $u = ax^2+cy^2+ez^2+2dyz+2czx+2bxy=0$; and if $y'^2 = 4z'x'$, P moves on a conic $v = 0$, and its polar touches

$$(ax + by + cz)(cx + dy + ez) = (bx + cy + dz)^2.$$

This conic is found to be the harmonic conic of u and v, and to meet v in the points which are the roots of the Hessian of u as a binary quartic. A binary transformation leaves v of the same form as before. The three quadratics correspond of course to right lines, and if a binary transformation make them the derived functions of a binary quartic u, they must be two tangents and their chord of contact to the harmonic conic, which is the reciprocal of v with respect to u. That the reducing cubic of the quartic is clear of its second term is (*Conics*, Art. 381, Ex. 1) equivalent to this special determination of the harmonic conic F. The conics u and v are said to be *apolar* (*Apolarität und rationale Curven*, F. Meyer, Tübingen, 1883).

Ex. 3. What relation holds among the coefficients of four quadratics if by the same linear transformation two of them become derived functions of one cubic and the other two of another?

$$\begin{vmatrix} a_1, & b_1, & a_2, & b_2 \\ b_1, & c_1, & b_2, & c_2 \\ a_3, & b_3, & a_4, & b_4 \\ b_3, & c_3, & b_4, & c_4 \end{vmatrix} = D_{11}D_{44} + D_{22}D_{33} - 2\{D_{12}D_{34} + D_{13}D_{42} - D_{14}D_{23}\},$$

the notation being the same as in Example 1, are easily found to be equal, and the condition required is got by equating them to zero.

Geometrically, the lines are such that a quadrilateral can be inscribed in the fundamental conic, having its sides each through the pole of one of the given lines; thus, points P and R can be found on the conic, such that when P is joined to the pole of line 2 and R joined to the pole of line 1 the lines meet in Q on the curve; and when P is joined to the pole of line 4 and R to the pole of line 3 these lines must meet in S on the curve.

Similarly considered, Ex. 1 has the lines so placed that a quadrilateral can be inscribed in the conic, having two alternate sides through the pole of one given line and the other sides severally through the poles of the other two given lines.

Ex. 4. What relation holds among the coefficients of three quadratics which can be the simplest quadric covariants H, H_1, H' of two cubics?

In Art. 219 it was pointed out that P^2, PQ, Q^2 can be independently expressed by means of the functions D. The values are in the present notation

$$P^2 = 4\,(4D_{13} - D_{22})$$

$$PQ = 8\,\{D_{12}D_{23} + D_{13}^2 - D_{11}D_{33} - D_{13}D_{22}\},$$

$$Q^2 = 16 \begin{vmatrix} D_{11}, & D_{21}, & D_{31} \\ D_{12}, & D_{22}, & D_{32} \\ D_{13}, & D_{23}, & D_{33} \end{vmatrix},$$

whence, comparing values of P^2Q^2, also writing the minors of the last determinant $A = D_{22}D_{33} - D_{23}^2$, &c., we find

$$4HF = B^2 + 2BG + CA.$$

The form of this relation, compared with that in Ex. 1, shows that the Jacobians of H, H_1, H' are second derived functions of a quartic.

Ex. 5. A cubic can be uniquely determined whose covariant H_1 with a given cubic vanishes identically.

Ex. 6. The P invariant of

$$u = (x - ay)\,(x - \beta y)\,(x - \gamma y), \quad v = A\,(x - ay)^3 + B\,(x - \beta y)^3 + C\,(x - \gamma y)^3,$$

vanishes identically. Similarly, $\overline{12}^n$ vanishes identically for two quantics

$$u = (x - a_1y)\,(x - a_2y)\ldots(x - a_ny), \quad v = A_1\,(x - a_1y)^n + \ldots + A_n\,(x - a_ny)^n.$$

Two binary quantics for which $\overline{12}^n$ vanishes are said to be *apolar* (see Ex. 2): also, *conjugate* binary forms by Dr. Schlesinger, *Math. Annalen* XXII.

Ex. 7. For two cubics, show that, in terms of the roots,

$$(\alpha - \alpha')\,(\beta - \beta')\,(\gamma - \gamma') + (\alpha - \beta')\,(\beta - \gamma')\,(\gamma - \alpha') + (\alpha - \gamma')\,(\beta - \alpha')\,(\gamma - \beta')$$

$$= (\alpha - \alpha')\,(\beta - \gamma')\,(\gamma - \beta') + (\beta - \beta')\,(\gamma - \alpha')\,(\alpha - \gamma') + (\gamma - \gamma')\,(\alpha - \beta')\,(\beta - \alpha') = \frac{3P}{aa'}.$$

Ex. 8. In the system of cubic and quadratic (Art. 198), calling the Jacobian of U and V, u, prove that the combinant P for the cubics u, U, is double the intermediate invariant I, and that the P for the cubics u, J, vanishes.

Ex. 9. For the cubicovariants J, J' of two cubics u, v the combinant

$$P(J, J') = 4D_{13}P - 2Q.$$

Ex. 10. Calculating, as in Art. 216, the determinant form $= R$ of the resultant of two cubics (Ex. 2, Art. 90), we get at once

$$\begin{aligned} -3R = {} & P(a_0a_4 - 4a_1a_3 + 3a_2^2) \\ & + (aa_3 - 3ba_2 + 3ca_1 - da_0)(a_1d' - 3a_2c' + 3a_3b' - a_4a') \\ & + (aa_4 - 3ba_3 + 3ca_2 - da_1)(a'a_3 - 3b'a_2 + 3c'a_1 - d'a_0) \\ = {} & \tfrac{1}{12}P^3 + \tfrac{2}{4}(L_1L'_2 - L_2L'_1) = \tfrac{1}{12}(P^3 - 27Q). \end{aligned}$$

For the biquadratic this gives the discriminant in the usual form, a_0 &c., are then coefficients of the Hessian.

Ex. 11. The expression $u^4D' - 4u^3vM' + 6u^2v^2N - 4uv^3M + v^4D$ contains as a factor the square of the Jacobian of u and v. If the division be effected, show that the quotient is $2\{P.J + 6H(J)\}$.

Ex. 12. The combinants P and Q of two cubics are each expressed in terms of the S and T invariants of the discriminant quartic, by a biquadratic whose quadrinvariant vanishes.

Ex. 13. If the roots of two cubics α, β, γ; α', β', γ', be connected by the relation $\sqrt[3]{\{(\alpha-\alpha')(\beta-\gamma')(\gamma-\beta')\}} + \sqrt[3]{\{(\beta-\beta')(\gamma-\alpha')(\alpha-\gamma')\}} + \sqrt[3]{\{(\gamma-\gamma')(\alpha-\beta')(\beta-\alpha')\}} = 0$, shew that $u + \lambda v$ admits of being a perfect cube.

Ex. 14. Determine the condition that a quartic may be such that its covariant $\overline{12}^2$ with another quartic may vanish identically.

It is $ST = 0$.

Ex. 15. The determinant form for J (Art. 202) is got by expressing that a quartic, whose covariant $\overline{12}^3$ with u vanishes identically, has a double root.

Ex. 16. If $a_0\dfrac{d}{da_1} + 2a_1\dfrac{d}{da_2} + \ldots + na_{n-1}\dfrac{d}{da_n} = \delta$, $b_0\dfrac{d}{db_1} + 2b_1\dfrac{d}{db_2} + \ldots + rb_{r-1}\dfrac{d}{db_r} = \delta'$, concomitants of the system $a_0x^n + \ldots = u$, $b_0x^r + \ldots = v$, can be derived from their sources by repeating the operation $\delta + \delta'$, and this operation on the coefficient of the highest power of x vanishes: as in the case of a single original function (Art. 147).

Ex. 17. If we combine any quadratic with a cubic and form their Jacobian Art. 198, it becomes in the method of Art. 219c a plane, and as we vary (the quadratics) the chords of the twisted cubic the Jacobian varies, but always passes through a fixed point. This point is easily seen by the determinant form for the cubicovariant of a cubic (Art. 195) to be the point which corresponds to that cubicovariant.

Ex. 18. In the system of cubic and quartic the simplest linear covariant p. 219 vanishes identically if the cubic be of the form uvw and the quartic $Au^4 + Bv^4 + Cw^4$.* In general, the system may be reduced to the form $Au^4 + Bv^4 + Cw^4$, $A'u^3 + B'v^3 + C'w^3$, where $u + v + w = 0$, by the canonizant $\begin{vmatrix} a', & a, & b, & y^3 \\ b', & b, & c, & -xy^2 \\ c', & c, & d, & x^2y \\ d', & d, & e, & -x^3 \end{vmatrix} = uvw = 0.$

LESSON XVIII.

HIGHER BINARY QUANTICS.

224. *The Quintic.* There are in all (including the quintic itself and four invariants) twenty-three forms. The invariants are J, K, L of the orders 4, 8, 12 respectively, and a skew invariant I of the 18th. The discriminant R of the quintic is not reckoned as a separate invariant, inasmuch as it is, as we shall presently see, a function $(J^3 - 128K)$ of the invariants J and K.

Three of the covariants invite special attention, viz. the Hessian $\overline{12}^2$, which if we take the quintic to be $(a,b,c,d,e,f\chi x,y)^5$, has for its value.

$$H=(ac-b^2)x^6+3(ad-bc)x^5y+3(ae+bd-2c^2)x^4y^2+(af+7be-8cd)x^3y^3$$
$$+3(bf+ce-2d^2)x^2y^4+3(cf-de)xy^5+(df-e^2)y^6.$$

There is a second covariant of the second order in the coefficients, viz. the covariant quadratic $\overline{12}^4$, the S of the quartic emanant, which has for its value

$$S=(ae-4bd+3c^2)x^2+(af-3be+2cd)xy+(bf-4ce+3d^2)y^2.$$

And thirdly, there is a covariant of the third order in the coefficients, viz. the canonizant, the determinant expression for which we gave Art. 169; that is to say, the covariant cubic which has for its factors the x, y, z of the canonical form. This covariant, which is also the T of the quartic emanant, has for its value

$$(ace-ad^2-eb^2+2bcd-c^3)x^3+(acf-ade-b^2f+bce+bd^2-c^2d)x^2y$$
$$+(adf-ae^2-bcf+bde+c^2e-cd^2)xy^2+(bdf-be^2-c^2f+2cde-d^3)y^3.$$

225. In studying the quintic we constantly use the canonical form $ax^5+by^5+cz^5$ (where $x+y+z=0$), to which it has been

shown (Art. 168) that the general equation may be reduced. For this form the three covariants just considered are respectively

$$H = bcy^3z^3 + caz^3x^3 + abx^3y^3,$$

$$S = bcyz + cazx + abxy,$$

$$T = abcxyz.$$

Differentiating the quintic with regard to x and y successively, we have $u_1 = ax^4 - cz^4$, $u_2 = by^4 - cz^4$. It is evident that the resultant of these two will be the discriminant of the quintic, and that the combinants of this system will be invariants of the quintic. These invariants are then immediately found from the expressions in Art. 223, where we must write for a and b, a and $-c$, for a' and b', b and $-c$. We have (24), and therefore $M = 0$; $(13) = 1$, $(12) = -1$, $(34) = -1$, $(14) = -1$, $(23) = 1$. We observe then at once that B vanishes. We can see, by counting constants, that any two cubics can be brought by linear transformation to be the two differentials of a single quartic; but two quartics cannot be similarly brought to be the differentials of a single quintic, unless the condition $B = 0$ be fulfilled. Or it may be otherwise stated that this is the condition that the quartics should be reducible to the form $au^4 + bv^4 + cw^4$, $a'u^4 + b'v^4 + c'w^4$.

The combinant A in like manner becomes

$$b^2c^2 + c^2a^2 + a^2b^2 - 2abc\,(a + b + c).$$

This, which we shall call J, is the simplest invariant of the quintic, and it may be obtained in other ways; viz. either by forming the discriminant of S, or the quadratic invariant $\overline{12}^6$ of H.

In either way we obtain the general value of $J =$

$$a^2f^2 - 10abef + 4acdf + 16ace^2 - 12ad^2e + 16b^2df$$
$$+ 9b^2e^2 - 12bc^2f - 76bcde + 48bd^3 + 48c^3e - 32c^2d^2.$$

226. The discriminant of the quintic may be obtained either from the theory of two quartics, or by direct elimination between the two differentials $ax^4 - cz^4$, $by^4 - cz^4$. When these

vanish together, we may take abc as the common value of ax^4, by^4, cz^4; whence $x=(bc)^{\frac{1}{4}}$, $y=(ca)^{\frac{1}{4}}$, $z=(ab)^{\frac{1}{4}}$. Substituting in $x+y+z=0$, we get the discriminant in the form

$$(bc)^{\frac{1}{4}}+(ca)^{\frac{1}{4}}+(ab)^{\frac{1}{4}}=0,$$

or $$\{b^2c^2+c^2a^2+a^2b^2-2abc(a+b+c)\}^2-128a^2b^2c^2(bc+ca+ab)=0.$$

Thus then we are led to the form for the discriminant J^2-128K, where K is the invariant of the eighth order in the coefficients, which for the canonical form is $a^2b^2c^2(bc+ca+ab)$.

This latter invariant may be otherwise defined: the invariant K is found by substituting in the usual way differential symbols for the variables, and operating with the square of the canonizant on the Hessian. This can easily be verified by the canonical form. Or else K can be found by forming the invariant I, as in Art. 198, of the covariant quadratic $bcyz+cazx+abxy$, and the canonizant. In any of these ways the general value is found to be $K=$

$$\begin{aligned}
&a^3cdf^3-a^3f^2ce^2-a^2f^3b^2d-3a^3f^2d^2e-3a^2f^3bc^2+5a^3fde^3+5af^3b^3c\\
&\quad-2a^3e^5-2b^5f^3+a^2f^2b^2e^2+11a^2f^2bcde-5a^2fbce^3-5af^2b^3de\\
&\quad+12a^2f^2bd^3+12a^2f^2c^3e-30a^2fbd^2e^2-30af^2b^2c^2e+15a^2bde^4\\
&\quad+15b^4cef^2-21a^2f^2c^2d^2-34a^2fc^2de^2-34af^2b^2cd^2+22a^2c^2e^4\\
&\quad+22b^4d^2f^2+78a^2fcd^3e+78af^2bc^3d-48a^2cd^2e^3-48b^3c^2df^2\\
&\quad-27a^2fd^5-27af^2c^5+18a^2d^4e^2+18b^2c^4f^2+133afb^2e^2cd\\
&\quad-54ab^2ce^4-54b^4de^2f-18afb^2d^3e-18afbc^3e^2+3ab^2d^2e^3\\
&\quad+3b^3c^2e^2f-220afbec^2d^2+106abc^2de^3+106b^3cd^2ef+93afbcd^4\\
&\quad+93afc^4de-30abe^2cd^3-30b^2ec^3df-9abed^5-9bec^5f-38ac^4e^3\\
&\quad-38b^3d^4f-42afc^3d^3+8ac^3d^2e^2+8b^2c^2d^3f+6ac^2d^4e+6bc^4d^2f\\
&\quad+27b^4e^4-81b^3e^3cd+38b^3e^2d^3+38b^2e^3c^3+25b^2e^2c^2d^2-57b^2ecd^4\\
&\quad-57be^2c^4d+18b^2d^6+18c^6e^2+74bec^3d^3-24bc^2d^5-24c^5d^2e+8c^4d^4.
\end{aligned}$$

The value of the discriminant in general can be derived hence, or else, as I originally obtained it, from the formula (Art. 220) for the resultant of two quartics. We thus find

$$R = a^4f^4 - 20a^3f^3be - 120a^3f^3cd + 160\,(a^3f^2ce^2 + a^2f^3b^2d)$$
$$+ 360\,(a^3f^2d^2e + a^2f^3bc^2) - 640\,(a^3fde^3 + af^3b^3c) + 256\,(a^3e^5 + b^5f^3)$$
$$- 10a^2f^2b^2e^2 - 1640a^2f^2becd + 320\,(a^2fbe^3c + af^2b^3ed)$$
$$- 1440a^2f^2\,(bd^3 + c^3e) + 4080\,(a^2fbe^2d^2 + af^2b^2ec^2)$$
$$- 1920\,(a^2be^4d + b^4ecf^2) + 2640a^2f^2c^2d^2 + 4480\,(a^2fc^2de^2 + af^2b^2cd^2)$$
$$- 2560\,(a^2c^3e^4 + b^4d^2f^2) - 10080\,(a^2fcd^3e + af^2bc^3d)$$
$$+ 5760\,(a^2cd^2e^3 + b^3c^2df^2) + 3456\,(a^2d^5f + af^2c^5)$$
$$- 2160\,(a^2d^4e^2 + b^2c^4f^2) - 180afb^3e^3 - 14920afb^2e^2cd$$
$$+ 7200\,(ab^2e^4c + b^4e^2df) + 960af\,(b^2ed^3 + be^2c^3)$$
$$- 600\,(ab^2e^3d^2 + b^3e^2c^2f) + 28480afbec^2d^2 - 16000(abe^3c^2d + b^3ecd^2f)$$
$$- 11520af\,(bcd^4 + c^4de) + 7200\,(abe^2cd^3 + b^2ec^3df)$$
$$+ 6400\,(ac^4e^3 + b^3d^4f) + 5120afc^3d^3 - 3200\,(ae^2c^3d^2 + b^2c^2d^3f)$$
$$- 3375b^4e^4 + 9000b^3e^3cd - 4000\,(b^3e^2d^3 + b^2e^3c^3) + 2000b^2e^2c^2d^2.$$

The discriminant may also be expressed as follows: Let

$$A = a^2f^2 - 34afbe + 76afcd - 32ace^2 - 32b^2df - 12aed^2 - 12bc^2f$$
$$+ 225b^2e^2 - 820becd + 480\,(bd^3 + c^3e) - 320c^2d^2;$$

$$B = 3a^2f^2 - 22afbe - 12afcd + 64\,(ace^2 + b^2df)$$
$$- 36\,(aed^2 + bc^2f) - 45b^2e^2 + 20becd;$$

$$C = a^2fe + 2afbd - 9abe^2 - 9afc^2 + 32acde - 18ad^3 + 6b^2cf$$
$$- 15bec^2 + 10bcd^2,$$

$$D = 3a^2df - 2a^2e^2 - 9afbc + abed + 18ac^2e - 12acd^2 + 6b^3f$$
$$- 15b^2ec + 10b^2d^2;$$

and let C', D' be the functions complemental to C and D (where all these functions vanish if three roots be equal), then three times the discriminant is

$$AB + 64CC' - 64DD'.$$

227. Quintics have also an invariant of the twelfth degree, which may be most simply defined as the discriminant of the canonizant. For the canonical form for which the canonizant is $abcxyz$, this discriminant is $-a^4b^4c^4$. And, in general, this discriminant is $-L$, where the following is the value of L as calculated by M. Faà de Bruno. To save space in printing we

omit the complementary terms. Thus $(a^4c^2de^2f^3)$ stands for $a^4c^2de^2f^3 + a^3b^2cd^2f^4$.

$$L = a^4c^2d^2f^4 - 2\,(a^4c^2de^2f^3) + (a^4c^2e^4f^2) - 6\,(a^4cd^3ef^3) + 16(a^4cd^2e^3f^2)$$
$$- 14\,(a^4cde^5f) + 4\,(a^4ce^7) + 4\,(a^4d^5f^3) - 11\,(a^4d^4e^2f^2) + 10(a^4d^3e^4f)$$
$$- 3\,(a^4d^2e^6) + 4a^3b^2cde^2f^3 - 2\,(a^3b^2ce^4f^2) + 6\,(a^3b^2d^3ef^3)$$
$$- 16\,(a^3b^2d^2e^3f^2) + 14\,(a^3b^2de^5f) - 4\,(a^3b^2e^7) + 50a^3bc^2d^2ef^3$$
$$- 82\,(a^3bc^2de^3f^2) + 32\,(a^3bc^2e^5f) - 36\,(a^3bcd^4f^3) + 30\,(a^3bcd^3e^2f^2)$$
$$+ 30\,(a^3bcd^2e^4f) - 24\,(a^3bcde^6) + 28\,(a^3bd^5ef^2) - 50\,(a^3bd^4e^3f)$$
$$+ 22\,(a^3bd^3e^5) + 16\,(a^3c^4e^3f^2) + 22a^3c^3d^3f^3 + 50\,(a^3c^3d^2e^2f^2)$$
$$- 16\,(a^3c^3de^4f) - 16\,(a^3c^3e^6) - 54\,(a^3c^2d^4ef^2) - 46\,(a^3c^2d^3e^3f)$$
$$+ 60\,(a^3c^2d^2e^5) + 6\,(a^3cd^6f^2) + 70\,(a^3cd^5e^2f) - 56\,(a^3cd^4e^4)$$
$$- 18\,(a^3d^7ef) + 14\,(a^3d^6e^3) + a^2b^4e^4f^2 + 132a^2b^3cde^3f^2 - 50(a^2b^3ce^5f)$$
$$+ 14\,(a^2b^3d^3e^2f^2) - 60\,(a^2b^3d^2e^4f) + 30\,(a^2b^3de^6) - 168a^2b^2c^2d^2e^2f^2$$
$$+ 48\,(a^2b^2c^2de^4f) + 4\,(a^2b^2c^2e^6) + 48\,(a^2b^2cd^4ef^2) + 2\,(a^2b^2cd^3e^3f)$$
$$- 6\,(a^2b^2cd^2e^5) - 62\,(a^2b^2d^6f^2) + 90\,(a^2b^2d^5e^2f) - 39\,(a^2b^2d^4e^4)$$
$$- 112\,(a^2bc^4e^4f) - 82a^2bc^3d^3ef^2 + 170\,(a^2bc^3d^2e^3f) + 104\,(a^2bc^3de^5)$$
$$+ 108\,(a^2bc^2d^3f^2) + 42\,(a^2bc^2d^4e^2f) - 298(a^2bc^2d^3e^4) - 242\,(a^2bcd^6ef)$$
$$+ 294\,(a^2bcd^5e^3) + 72\,(a^2bd^8f) - 78\,(a^2bd^7e^2) + 164\,(a^2c^5de^3f)$$
$$+ 24\,(a^2c^5e^5) - 63a^2c^4d^4f^2 - 394\,(a^2c^4d^3e^2f) - 194\,(a^2c^4d^2e^4)$$
$$+ 324\,(a^2c^3d^5ef) + 440\,(a^2c^3d^4e^3) - 78\,(a^2c^2d^7f) - 428\,(a^2c^2d^6e^2)$$
$$+ 180\,(a^2cd^8e) - 27\,(a^2d^{10}) + 18ab^5e^5f - 38ab^4cde^4f + 36\,(ab^4ce^6)$$
$$+ 204\,(ab^4d^3e^3f) - 102\,(ab^4d^2e^5) - 308\,(ab^3c^2de^5) - 42ab^3c^2d^2e^3f$$
$$- 674\,(ab^3cd^4e^2f) + 590\,(ab^3cd^3e^4) + 128\,(ab^3d^6ef) - 138\,(ab^3d^5e^3)$$
$$+ 4\,(ab^2c^4e^5) + 652\,(ab^2c^3d^2e^4) + 714ab^2c^3d^3e^2f + 498\,(ab^2c^2d^5ef)$$
$$- 1246\,(ab^2c^2d^4e^3) - 224\,(ab^2cd^7f) + 516\,(ab^2cd^6e^2) - 48\,(ab^2d^8e)$$
$$- 136\,(abc^5de^4) - 1078abc^4d^4ef - 206\,(abc^4d^3e^3) + 342\,(abc^3d^6f)$$
$$+ 804\,(abc^3d^5e^2) - 506\,(abc^2d^7e) + 90\,(abcd^9) - 16\,(ac^7e^4)$$
$$+ 220\,(ac^6d^2e^3) - 106ac^5d^5f - 392\,(ac^5d^4e^2) + 222(ac^4d^6e) - 40(ac^3d^8)$$
$$- 27b^6e^6 + 234b^5cde^5 - 32\,(b^5d^3e^4) - 713b^4c^2d^2e^4 + 246\,(b^4cd^4e^3)$$
$$- 4\,(b^4d^6e^2) + 866b^3c^3d^3e^3 - 550\,(b^3c^2d^5e^2) + 56\,(b^3cd^7e) + 4\,(b^3d^9)$$
$$- 139b^2c^4d^4e^2 + 354\,(b^2c^3d^6e) - 83\,(b^2c^2d^8) - 330bc^5d^5e$$
$$+ 72\,(bc^4d^7) - 16c^6d^6.$$

On inspecting this invariant it will be seen that it vanishes if b, c, d all vanish. Consequently the form $ax^5 + 5exy^4 + fy^5$, to which Mr. Jerrard has shown that the quintic can be brought by a non-linear transformation, is one to which no quintic can be brought by linear substitution unless $L = 0$.

228. We take J, K, L as the fundamental invariants of the quintic, and we proceed to show how all its other invariants can be expressed in terms of these. In the first place, it will be observed that the interchange in the canonical form either of x and y, or of x and z, is a linear transformation whose modulus is -1. Hence, if any invariant is such that when transformed it is multiplied by an even power of the modulus of transformation, it must, for the canonical form, be unaltered by any interchange of a, b, c; that is to say, it must be a symmetric function of these quantities. If the invariant is multiplied by an odd power of the modulus, it must be such as to change sign when any two of the quantities a, b, c are interchanged; it must therefore be of the form $(a-b)(b-c)(c-a)$ multiplied by a symmetric function of a, b, c. Now an invariant is in transformation multiplied by a power of the modulus equal to its weight. And (Art. 143) the weight of an invariant of the quintic, whose order is n, is $\frac{5}{2}n$. A quintic cannot have an invariant of odd order in the coefficients. If the order is a multiple of 4, the weight is an even number, and the sign of the invariant is unaltered by the interchange of x and y. If the order be not divisible by 4, the invariant is what we have called *skew*, that is to say, such as to change sign when x and y are interchanged. Let us first examine the former kind, which we have seen must, for the canonical form, be symmetric functions of a, b, c. Now, since $J = (bc + ca + ab)^2 - 4abc(a + b + c)$, $K = a^2b^2c^2(bc + ca + ab)$, $L = a^4b^4c^4$, (from which we infer $H = \frac{1}{4}(K^2 - JL) = a^5b^5c^5(a + b + c)$,*) it follows that if we are

* The reader must be careful to observe that though, in the case of the canonical form, $a^5b^5c^5(a + b + c)$, for example, is divisible by $a^4b^4c^4$, we have no right to infer that in general H is divisible by L, unless in cases where the quotient $abc(a + b + c)$ has been also proved to be an invariant.

given any quintic, and transform it to the canonical form by a substitution whose modulus is unity, the numerical values of the new a, b, c are given by the cubic

$$\alpha^3 - \frac{H}{L^{\frac{2}{3}}}\alpha^2 + \frac{K}{L^{\frac{1}{3}}}\alpha - L^{\frac{1}{3}} = 0.$$

Now the order of any symmetrical function of a, b, c will be equal to its weight in the coefficients of this cubic, and when this weight is a multiple of 4, it is easy to see that the symmetric function is a rational function of J, K, L.

Being given, therefore, any invariant whose order in the coefficients is a multiple of 4, it has been proved that we can write down a rational function of J, K, L, which, for the canonical form, shall have the same value as this invariant, and therefore be always identical with it. And since it would be manifestly absurd to suppose an integral function of the coefficients to be equal to an irreducible fraction, it follows that every non-skew invariant is an integral function of J, K, L.

If we make the first three coefficients a, b, c each equal 0, J, K, L all vanish. Hence when three roots of a quintic are all equal, these three invariants vanish.* If we make a, b, e, f all equal 0, J becomes $-32c^2d^2$, and L, $-16c^6d^6$, and therefore $J^3 - 2048L$ vanishes. Quintics therefore which have two pairs of equal roots must not only have the discriminant $= 0$, but also $J^3 = 2048L$.

229. The simplest skew invariant is got by forming the resultant of the quintic $ax^5 + by^5 + cz^5$, and its canonizant $abcxyz$. Substituting successively the three roots of the canonizant in the quintic, and multiplying together, we get for the resultant $a^5b^5c^5(b-c)(c-a)(a-b)$. This invariant, therefore, is of the eighteenth order. Previous to its discovery by M. Hermite,†

* In general all the invariants of a quantic vanish, if more than $\frac{1}{2}n$ of its roots be all equal. For when half the coefficients, counting from one end, simultaneously vanish, no term of the proper weight (Art. 143) can be made with the remaining coefficients.

† See *Cambridge and Dublin Mathematical Journal*, vol. IX. p. 172. M. Hermite works with a new canonical form, the x and y of which are the two factors of the quadratic covariant. The quintic then is supposed to be such that $ae - 4bd + 3c^2$, $bf - 4ce + 3d^2$ both vanish, and the quadratic covariant reduces to xy. The advantage of this is that the operating symbol thence derived is simply $\frac{d^2}{dx\,dy}$, and some of the covariants

the possibility of the existence of skew invariants had not been recognised. I took the trouble to calculate this invariant, and the result is printed (*Philosophical Transactions*, 1858, p. 455*), but as it consists of nearly nine hundred terms I cannot afford room for it here. The leading terms are $a^7d^5f^6 - a^6c^5f^7$; in this, as in every skew invariant, the complementary terms having opposite signs, and the symmetrical terms vanishing. Moreover if the alternate terms in any equation be wanting every skew invariant vanishes. For in this case the weight of each coefficient is even, but the weight of any skew invariant is an odd number. Thus I vanishes if b, d, f vanish; that is to say, if the quintic can be reduced to the form $x(x^2-\alpha^2)(x^2-\beta^2)$, in other words, if we consider the quintic as denoting five points on a right line, the vanishing of I is the condition that one of these points should be a self-conjugate point of the involution determined by the other four.

By the argument used, it is proved that every skew invariant of a quintic must be the product of this invariant I by a rational function of J, K, L.

230. The square of I being of the thirty-sixth degree can be expressed rationally in terms of J, K, L (Art. 228). The actual expression is easily found.

By forming the discriminant of the cubic (Art. 228)

$$\alpha^3 - \frac{H}{L^{\frac{2}{3}}}\alpha^2 + \frac{K}{L^{\frac{1}{3}}}\alpha - L^{\frac{1}{3}},$$

we obtain the product of the squares of the differences of a, b, c in terms of J, K, L, and thus have

$$I^2L = H^2K^2 + 18HKL^2 - 27L^4 - 4K^3L^2 - 4H^3;$$

or putting for H its value $\frac{1}{4}(K^2 - JL)$, and dividing by L, we have

$$16I^2 = JK^4 + 8LK^3 - 2J^2LK^2 - 72JKL^2 - 432L^3 + J^3L^2.$$

In the last equation of Art. 222 when we make $B=0$ for two quartics derived from a quintic we find by the same article

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obtained by thus differentiating assume a very simple form. Notwithstanding, I have preferred using Sylvester's canonical form, which I find much more convenient.

* Where the coefficient of $b^7d^7e^3f$ should have been printed 12500.

$E=D$ and by Art. 220 $I+C=0$; whence M^2 becomes the function written here.

231. We come now to the covariants. We have already (Art. 224) mentioned the quadratic covariant S and the cubic covariant T. Considering this system of a cubic and quadratic, we have (Art. 198) a series of covariants which give completely all the covariants of the quintic which are not higher than the third order in the variables. The five invariants of Art. 198 reduce to four J, K, L, I already mentioned, the discriminant of the cubic, and the resultant of cubic and quadratic, both reducing to L. The four linear covariants of the system of cubic and quadratic give four linear covariants of the quintic, of the orders 5, 7, 11, 13, which for the canonical form are respectively

$$abc\,(bcx+cay+abz),$$

$$abc\,\{(b^2c^2+a^2bc)\,(y-z)+(c^2a^2+b^2ac)\,(z-x)+(a^2b^2+c^2ab)\,(x-y)\},$$

$$a^3b^3c^3\,\{bc\,(y-z)+ca\,(z-x)+ab\,(x-y)\},$$

$$a^4b^4c^4\,\{ax+by+cz\}.$$

These are the only distinct linear covariants of the quintic. If we eliminate either between the first and last of these, or between the second and third, or between the first of them and the canonizant, we get Hermite's I; and if between the first linear covariant and the quintic itself we get $I\,(J^2-3K)$. Thus, then, if I vanish, or if $J^2=3K$, the quintic is immediately soluble, one of the roots being given by that linear covariant. Hermite has studied the quintic by transforming the equation, so as to take the first two linear covariants for x and y, when all the coefficients in the transformed equation are found to be invariants. The transformation becomes impossible when the two linear covariants are identical, which will be when their resultant $JK+9L$ vanishes.

The system of cubic and quadratic have (Art. 198) three quadratic covariants, viz. in addition to S itself, the Hessian of T or $a^2b^2c^2\,(x^2+y^2+z^2)$, and the Jacobian of this and S, or

$$a^2b^2c^2\,\{bcx\,(y-z)+cay\,(z-x)+abz\,(x-y)\}.$$

These are the only distinct quadratic covariants of the quintic.

Lastly, there are three cubic covariants, viz. in addition to T itself, its cubic covariant $a^3b^3c^3(y-z)(z-x)(x-y)$; and the Jacobian of S and T,

$$abc\{bcyz(y-z)+cazx(z-x)+abxy(x-y)\}.$$

These are the only cubic covariants of the quintic. We have now enumerated fourteen forms, whose order in the variables is not higher than the third; adding to these the quintic and its Hessian, there are still seven forms to be mentioned. If we operate with S upon H, we get a quartic of the fourth order in the coefficients, which only differs by a multiple of the square of S from $abc(ax^4+by^4+cz^4)$.

A second quartic covariant is the Jacobian of this and S, or

$$abc\{a^2(b-c)x^4+b^2(c-a)y^4+c^2(a-b)z^4\}.$$

These are the only two quartic covariants. We have a quintic covariant by taking the Jacobian of S and U, viz.

$$a^3(b-c)x^5+b^3(c-a)y^5+c^3(a-b)z^5$$
$$-abc(y-z)(z-x)(x-y)(yz+zx+xy).$$

A second quintic covariant is found by taking the Jacobian of U and the quadratic covariant $a^2b^2c^2(x^2+y^2+z^2)$. This gives

$$a^3b^2c^2\{ax^4(y-z)+by^4(z-x)+cz^4(x-y)\}.$$

Of sextic forms there only is, in addition to the Hessian, the Jacobian of S and H, or

$$abc\{ax^5(y-z)+by^5(z-x)+cz^5(x-y)\}.$$

There is one septic form, viz. the Jacobian of U and the simplest quartic covariant, or

$$abc\{bcy^3z^3(y-z)+caz^3x^3(z-x)+abx^3y^3(x-y)\}.$$

And lastly, one nonic, namely, the Jacobian of U and H, or

$$a^3bx^7y^2-a^3cx^7z^2+b^3cy^7z^2-b^3ay^7x^2+c^3az^7x^2-c^3bz^7y^2$$
$$+abcx^2y^2z^2(y-z)(z-x)(x-y).$$

232. The forms might also have been arranged, as Prof. Cayley has done, according to their order in the coeffi-

cients. We give here, in his order, the leading terms of the less complicated.

(1) u, Quintic, a.

(2) S, Quadratic, $ae - 4bd + 3c^2$.

(3) H, Sextic, $ac - b^2$.

(4) T, Cubic, $ace - ad^2 - b^2e + 2bcd - c^3$.

(5) Quintic, $a^2f - 5abe + 2acd + 8b^2d - 6bc^2$.

(6) Nonic, $a^2d - 3abc + 2b^3$.

(7) Invariant J already given; fourth degree in coefficients.

(8) Quartic, $a^2(e^2 - df) + a(3bcf - 3bde - 4c^2e + 4cd^2)$
$+ 5b^2ce + 2b^2d^2 - 2b^3f - 8bc^2d + 3c^4$.

This differs by the square of S from the corresponding quartic covariant, Art. 231; and is $\overline{12}^2$ of T and u.

(9) Sextic, $a^2(cf - de) - ab^2f - 2abce + 4abd^2 - ac^2d + 3b^2e$
$- 6b^2cd + 3bc^3$.

(10) Linear, $a^2(cf^2 - 2def + e^3) + a(-b^2f^2 - 4bcef + 8bd^2f)$
$+ a(-2bde^2 - 2c^2df + 14c^2e^2) + a(-22cd^2e + 9d^4) + 6b^3ef - 12b^2cdf$
$- 15b^2ce^2 + 10b^2d^2e + 6bc^2f + 30bc^2de - 20bcd^3 - 15c^4e + 10c^3d^2$.

(11) Cubic, $a^2(cef - 3d^2f + 2de^2) + a(-b^2ef + 14bcdf - 11bce^2)$
$+ a(-bd^2e - 9c^2f + 14c^2de - 6cd^3) - 8b^3df + 9b^3e^2 + 6b^2c^2f - 16b^2cde$
$+ 8b^2d^3 + 3bc^3e - 2bc^2d^2$. This is the Jacobian of S and T.

(12) Septic, $a^2(2c^2f - 5cde + 3d^3) + a(-4b^2cf + 5b^2de + 5bc^2e)$
$+ a(-7bcd^2 + c^3d) + 2b^4f - 5b^3ce - 2b^3d^2 + 8b^2c^2d - 3bc^4$.

(13) Quadratic, $a^2(-c^2f^2 + 5cdef - 3ce^3 - 3d^3f + 2d^2e^2)$
$+ a(2b^2cf^2 - 5b^2def + 3b^2e^3 - 5bc^2ef + 7bcd^2f)$
$+ a(-bcde^2 - bd^3e - c^3df + 6c^3e^2 - 8c^2d^2e + 3cd^4)$
$- b^4f^2 + 5b^3cef + 2b^3d^2f - 3b^3de^2 - 8b^2c^2df - 4b^2c^2e^2 + 7b^2cd^2e$
$- b^2d^4 + 3bc^4f + 5bc^3de - 4bc^2d^3 - 3c^5e + 2c^4d^2$.

(14) Quartic, $a^3(-df^2 + e^2f) + a^2(3bcf^2 + 2bdef - 5be^3 - 8c^2ef)$
$+ a^2(2cd^2f + 12cde^2 - 6d^3e) + a(-2b^2f^2 - 2b^2cef - 6b^2d^2f + 13b^2de^2)$
$+ a(20bc^2df + 4bc^2e^2 - 52bcd^2e + 24bd^4 - 9c^4f + 20c^3de - 10c^2d^3)$
$+ 6b^4ef - 12b^3cdf - 15b^3ce^2 + 10b^3d^2e + 6b^2c^3f + 30b^2c^2de$
$- 20b^2cd^3 - 15bc^4e + 10bc^3d^2$.

(15) Linear, $a^3(cf^3 - 4def^2 + 3e^3f) + a^2(-b^2f^3 - 3bcef^2)$
$+ a^2(16bd^2f^2 + 4bde^2f - 15be^4 - 6c^2df^2 + 4c^2e^2f - 22cd^2ef)$
$+ a^2(26cde^3 + 9d^4f - 12d^3e^2) + a(7b^3ef^2 - 30b^2cdf^2 + b^2ce^2f)$
$+ a(-74b^2d^2ef + 84b^2de^3 + 18bc^2f^2 + 160bc^2def - 98bc^2e^3)$
$+ a(-20bcd^3f - 94bcd^2e^2 + 51bd^4e - 81c^4ef + 18c^3d^2f + 140c^3de^2)$
$+ a(-100c^2d^3e + 18cd^5) + 8b^4df^2 - 18b^4e^2f - 6b^3c^2f^2 + 32b^3cdef$
$+ 45b^3ce^3 + 112b^3d^3f - 150b^3d^2e^2 - 6b^2c^3ef - 284b^2c^3d^2f + 50b^2c^2de^2$
$+ 320b^2cd^3e - 120b^2d^5 + 216bc^4df - 15bc^4e^2 - 310bc^3d^2e + 130bc^2d^4$
$- 54c^6f + 90c^5de - 40c^4d^3.$

(16) Quintic, $a^3(cdf^2 - 2ce^2f + 2d^2ef - de^3) + a^2(-b^2df^2 + 2b^2e^2f)$
$+ a^2(-3bc^2f^2 - 6bcdef + 13bce^3 - 8bd^3f + 2bd^2e^2)$
$+ a^2(16c^3ef - 2c^2d^2f - 38c^2de^3 + 34cd^3e - 9d^5)$
$+ a(5b^3cf^2 + 2b^3def - 12b^3e^3 - 24b^2c^2ef + 52b^2cd^2f + 7b^2cde^2)$
$+ a(-22b^2d^3e - 52bc^3df + 34bc^3e^2 + 8bc^2d^2e - bcd^4 + 18c^5f)$
$+ a(-25c^4de + 10c^3d^3) - 2b^5f^2 + 10b^4cef - 28b^4d^2f + 30b^4de^2$
$+ 32b^3c^2df - 35b^3c^2e^3 - 50b^3cd^2e + 30b^3d^4 - 12b^2c^4f + 70b^2c^3de$
$- 40b^2c^2d^3 - 15bc^5e + 10bc^4d^2.$

(17) Invariant K already given, 8th degree in coefficients.

(18) Quadratic, 8th degree in coefficients.

(19) Cubic, 9th degree in coefficients.

(20) Linear, 11th in coefficients.

(21) Invariant L already given, 12th degree in coefficients.

(22) Linear, 13th in coefficients.

(23) Invariant I, 18th in coefficients.

For (18), (19), (20), (22) we refer to Prof. Cayley's Ninth Memoir on Quantics, *Phil. Trans.*, 1871, p. 17.

233. Prof. Cayley* has been led to consider in the theory of the quintic a new canonical form, which is obtained as follows: Taking for convenience the quintic to be

$$(a, b, c, d, e, f\chi x, y)^5,$$

* It has been already mentioned (p. 134) that the method of discussing covariants by means of their leading terms or *sources* was introduced by Prof. M. Roberts See *Quarterly Journal*, vol. IV.

using small Roman letters for the coefficients, suppose in the first instance that a, b, c, d, e, f denote the leading coefficients of the first six covariants of Art. 232 respectively, thus

$$a = a, \qquad c = ac - b^2, \qquad e = a^2f - 5abe + 2acd + 8b^2d - 6bc^2,$$
$$b = ae - 4bd + 3c^2, \quad d = ace - ad^2 - b^2e - c^3 + 2bcd, \quad f = a^2d - 3abc + 2b^3,$$

where $-f^2 = a^2(ad - bc) + 4c^3$ identically, so that any rational and integral function containing f can always be expressed as a function linear in regard to f. This being so, we have the function

$$\frac{1}{a}(a, b, c, d, e, f)(x - by, ay)^5 = x^5 + 10x^3y^2(ac - b^2) + 10x^2y^3(a^2d - 3abc + 2b^3)$$
$$+ 5xy^4(a^3e - 4a^2bd + 6ab^2c - 3b^4) + y^5(a^4f - 5a^3be + 10a^2b^2d - 10ab^3c + 4b^5),$$

and forming the values of $a^2b - 3c^2$ and $a^2e - 2cf$, this is found to be

$$= (1, 0, c, f, a^2b - 3c^2, a^2e - 2cf)(x, y)^5.$$

The last-mentioned function, considering therein a, b, c, e, f as denoting not the leading coefficients, but the covariants themselves, and (x, y) as variables distinct from those of the quintic and its covariants, is the canonical form in question. Using in like manner d to stand for the covariant, we have between the covariants a, b, c, d, e, f the foregoing identical equation

$$-f^2 = a^2(ad - bc) + 4c^3,$$

which is to be used to reduce functions of the covariants so as to render them linear in regard to f.

234. *Criteria for the reality of roots of quintics.* It ought to have been stated earlier that the sign of the discriminant of any quantic enables us at once to determine whether it has an even or odd number of pairs of imaginary roots. Imagine the quantic resolved into its real quadratic factors, then (Art. 110) the discriminant of the quantic is equal to the product of the discriminants of all the quadratics, multiplied by the square of the product of the resultants of every pair of factors. These resultants are all real, and their squares positive; therefore, in considering the sign of the discriminant, we need only attend to the discriminants of the quadratic factors. But the square of the difference of the roots of a quadratic is positive when the roots are real, and negative when they are

imaginary. It follows then that the product of the squares of the differences of the roots of any quantic is positive when it has an even number of pairs of imaginary roots, and negative when it has an odd number. We have been accustomed to write the discriminant giving the positive sign to the term which is a power of the product of the two extreme coefficients. This will have the same sign as the product of the squares of differences of the roots when the order of the quantic is of the form $4m$ or $4m+1$, and the opposite sign when the order is of the form $4m+2$ or $4m+3$. We see then, in the case of the quintic, that if the discriminant be positive, there will be either four imaginary roots or none; and if the discriminant be negative, there will be two imaginary roots. It remains then further to distinguish the cases when all the roots are real, and where only one is so.

235. In order to discriminate between these remaining cases, there are various ways in which we may proceed. The following* are, in their simplest forms, the criteria furnished by Sturm's theorem. Let J be the quartic invariant as before, and

$$H = b^2 - ac, \quad S = ae - 4bd + 3c^2, \quad T = ace + 2bcd - ad^2 - eb^2 - c^3,$$

$$M = a^2e^2 - a^2df + 3abcf - 3abde + 4acd^2 - 4ac^2e - 2b^3f + 5b^2ce + 2b^2d^2 - 8bc^2d + 3c^4,$$

then the leading terms in the Sturmian functions are proportional to a, a, H, $5HS + 9aT$, $-HJ + 12SM + 4S^3 - 216T^2$, the last of course being the discriminant; and the conditions furnished by Sturm's theorem to discriminate the cases of four and no imaginary roots, are that when all the roots are real the three quantities H, $5HS + 9aT$, $-HJ +$ &c. must all be positive.

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* These values are given by Mr. M. Roberts, *Quarterly Journal*, vol. IV. p. 175. The reader who may use Prof. Cayley's tables of Sturmian functions (*Philosophical Transactions*, vol. CXLVII. p. 735) must be cautioned that the fourth and fifth functions are there given with wrong signs.

M is already written as (8) in Art. 232, and is connected with D in Art. 226 by the equation $D = S^2 - 3M$. In fact the expression for the discriminant R there given is $9R = 25J^2 - 192\,(D_0D_4 - 4D_1D_3 + 3D_2^2)$, where the covariant whose source is D is written $D_0x^4 + 4D_1x^3y +$ &c., and $D = D_0$, $C = 2D_1$, $3B + A = -10J$, $3B - A = 48D_2$, $C' = 2D_3$, $D' = D_4$.

236. We may apply these conditions to the canonical form

$$(c-a)\,x^5+5cx^4y+10cx^3y^2+10cx^2y^3+5cxy^4+(c-b)\,y^5,$$

in which case the equality of all but two of the coefficients renders the direct calculation also easy. We easily find then that the constants are $c-a$, $c-a$, ac, $-a^2c^2$; and the fourth being essentially negative, we need not proceed further, and we learn that the equation just written has always imaginary roots. We find then that when the invariant L of a quintic is positive, the roots of the equation cannot be all real. For L being, with sign changed, the discriminant of the canonizant, when L is positive, the roots of the canonizant are all real, and the quintic can be brought to the canonical form by a real transformation.

When L is negative, two factors of the canonizant are imaginary, and the canonical form is

$$a\,(-2x)^5+\{c-d\sqrt{(-1)}\}\,\{x+y\sqrt{(-1)}\}^5 + \{c+d\sqrt{(-1)}\}\,\{x-y\sqrt{(-1)}\}^5,$$

which, expanded, is

$$dy^5+5cy^4x-10dy^3x^2-10cy^2x^3+5dyx^4+(c-16a)\,x^5.$$

Writing for brevity $c^2+d^2=r^2$, I find for this form the Sturmian constants to be d, d, r^2, r^4, $r^2\,(-4a^2d^2+20acr^2+5r^4)$, and it would seem that the discriminant being positive, the roots are all real if d and $-4a^2d^2+20acr^2+5r^4$ are both positive.*

237. In practice the criteria† furnished by Sturm's theorem are more convenient than any other, because the functions to be calculated are of lower order in the coefficients. It is, however, theoretically desirable to express these criteria in terms of the invariants, and this is what has been effected by different

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* I give this result, though suspecting its accuracy, because it seems to me to disagree with the theory derived from the other methods.

† It may be noticed that there is no difficulty in writing down a multitude of criteria which *might* indicate the existence of imaginary roots; for any symmetric function of squares of differences of roots $\Sigma\,(\alpha-\beta)^2$, &c. must be positive if all the roots are real. We can without difficulty write down such functions which are also invariants; and which, if negative, show that the equation has imaginary roots. But then these may also be positive when the roots are imaginary, and the problem is to find some criterion or system of criteria, some one of which *must* fail to be satisfied when the roots are not all real.

methods by Hermite and by Sylvester. We proceed briefly to explain the principles of Sylvester's method, which is highly ingenious. We have seen already that when the invariants J, K, L are given, then a, b, c of the canonical form may be determined by a cubic equation; and we can infer that to every given system of values of J, K, L will correspond some quintic. But to every system of values of J, K, L will not correspond a *real* quintic. In fact, we have seen, Art. 230, that the J, K, L of every quintic with real coefficients, are such that the quantity G is essentially positive, where G is

$$JK^4 + 8LK^3 - 2J^2LK^2 - 72JL^2K - 432L^3 + J^3L^2.$$

For G has been shown to be the perfect square of a real function of the coefficients of the general quintic, viz. $a^7d^5f^6$+&c., this being the eliminant of the quintic and its canonizant, and therefore necessarily real. We may in the above substitute for K its value in the discriminant from the equation $J^2 - 128K = D$, and so write G,

$$JD^4 - 4(J^3 + 2^8L)D^3 + (6J^3 - 29\cdot 2^{10}L)J^2D^2$$
$$- 4(J^6 - 61.2^8J^3L - 9.2^{22}L^2)JD + (J^3 - 2^{11}L)^2(J^3 - 27.2^{10}L).$$

If now, to assist our conceptions, we take J, D, L for the coordinates* of a point in space, then $G=0$ represents a surface; and points on one side of it, making G positive, answer to real quintics, while points on the other side, making G negative†, answer to quintics with imaginary coefficients.

238. Now, in the next place, we say that if the coefficients in an equation be made to vary continuously, the passage from real to imaginary roots must take place through equal roots. For, let any quantic $\phi(x)$ become by a small change of coefficients $\phi(x) + \varepsilon\psi(x)$, where ε is infinitesimal, and let α be a real root of the first, $\alpha + h$ a root of the second; then we

* Sylvester takes L in the usual direction of x, J of y, and D of z.

† Points for which $G = 0$ answer to real quintics, and it is easy to see that in this case the equation admits of linear transformation to the recurring form. For we have proved that when $G = 0$ two of the coefficients of the canonical form are equal. The equation is therefore of the form $ax^5 + ay^5 + b(x + y)^5 = 0$.

have $\phi(\alpha+h)+\varepsilon\psi(\alpha)=0$; whence, since $\phi(\alpha)=0$, we have $h\phi(\alpha)+\varepsilon\psi(\alpha)=0$, which gives a real value for h. The consecutive root $\alpha+h$ is therefore also real. But if $\phi'(\alpha)$ vanishes as well as $\phi(\alpha)$, the lowest term in the expansion of $\phi(\alpha+h)$ will be h^2, and the value of h may possibly be imaginary. When, therefore, the original quantic has equal roots, the corresponding roots of the consecutive quantic may be imaginary.

It follows then, that if we represent systems of values of J, D, L by points in space, in the manner indicated in the last article, two points will correspond to quintics having the same number of real roots, provided that we can pass from one to the other without crossing either the plane D or the surface G. If points lie on opposite sides of the plane D, we evidently cannot pass from one to the other without having, at an intervening point, $D=0$, at which point a change in the character of the roots might take place. If two points, both fulfilling the condition G positive, be separated by sheets of the surface G, we can not pass continuously from one of the corresponding quintics to the other; because when on crossing the surface we have G negative, the corresponding quintic has imaginary coefficients. But when two points are not separated in one of these ways, we can pass continuously from one to the other, without the occurrence of any change in the character of the corresponding quintics.

Now Sylvester's method consists in shewing, by a discussion of the surface G, that all points fulfilling the condition G positive, which he calls *facultative* points, may be distributed in three blocks separated from each other either by the plane D or the surface G. And since there may evidently be quintics of three kinds, viz. having four, two, or no imaginary roots, the points in the three blocks must correspond respectively to these three classes. I have not space for the elaborate investigation of the surface G, by which Sylvester establishes this; but the following is sufficient to enable the reader to convince himself of the truth of his conclusions.

239. One of the three blocks we may dispose of at once, viz. points on the negative side of the plane D, which we have

seen (Art. 234) correspond to quintics having two imaginary roots. Next with regard to points for which D is positive. We have seen, in the last article, that a change in the character of the roots takes place only when $D=0$; our attention is therefore directed to the section of G by the plane D. We see at once, by making $D=0$ in the value of G (Art. 237), that the remainder has a square factor, and consequently that the surface G touches D along the curve $J^3-2^{11}L$, and cuts it along $J^3-27.2^{10}L$. Now, if a surface merely cut a plane, the line of section is no line of separation between points on the same side of the surface. If, for example, we put a cup on a table, there is free communication between all the points inside the cup and between all those outside it. But if a plane touch a surface, as, for instance, if we place a cylinder on a table, then while there is still free communication between the points inside the cylinder, the line of contact acts as a boundary line, cutting off communication as far as it extends, between points outside the cylinder on each side of the boundary.

Now Sylvester's assertion is, that if we take the negative quadrant, viz. that for which both J and L are negative, and if we draw in the plane of xy the curve $J^3-2^{11}L$, then all facultative points in that quadrant, lying above the space included between the curve and the axis $L=0$, form a block completely separated from the rest, and correspond to the case of five real roots.

240. In order to see the character of the surface, I form the discriminant of G considered as a function of K, which I find to be $-L^7(J^3+27L)^3$. Consequently, when both J and L are negative, the discriminant is negative, and the equation in K has only two real roots. To every system of values, therefore, of J and L correspond two values of K, and consequently two values of D and the surface is one of two sheets. Now I say that it is the space between these sheets for which G is positive. In fact, since G is JD^4+&c., it may be resolved into its factors $J(D-\alpha)(D-\beta)\{(D-\gamma)^2+\delta^2\}$; and since J is supposed to be negative in the space under consideration, D must evidently be intermediate between α and β in order that G should be positive.

Now the last term of the equation being $(J^3-2^{11}L)^2(J^3-27.2^{10}L)$, if J^3 be nearly equal to $2^{11}L$, will be of opposite sign to L, or in the present case will be positive. And the coefficient of D^4 being negative, we see that on both sides of the line $J^3=2^{11}L$ the values of D are, one positive and the other negative, that is to say, the two sheets of the surface are one above and the other below the plane D. But I say it is the upper sheet which touches D along $J^3-2^{11}L$. This may be seen immediately by looking at the sign of the penultimate term in the equation for D, by which we see that when the last term vanishes, the two roots are 0 and negative. The theory then already explained shows that the curve $J^3=2^{11}L$ acts as a boundary line cutting off communication in that direction between facultative points on the upper side of D. But, again, communication in the other direction is cut off by the plane $L=0$. For when we make L positive, the discriminant becomes positive, and the equation in D has either four real or four imaginary roots. But the first Sturmian constant is proportional to $L(J^3+12L)$, which, when J is negative, and L positive and small, is negative. Immediately beyond the plane L, therefore, the equation to determine D has four imaginary roots, or the surface does not exist. The facultative points, therefore, lying as they do *within* the surface or between its sheets, are cut off by the plane L, on which the sheets unite, from communication with points beyond it. Thus the isolation of the block under consideration has been proved.

I need not enter into equal detail to prove that all other facultative points have free communication *inter se*. The line of contact $2^{11}L-J^3$ is no line of separation in the quadrant where J and L are both positive. For then it is seen, as before, that it is the points *outside* the two sheets which are facultative, and not the points between the surface and touching plane.

The result of this investigation is, that in order to have all the roots real, we must have the quantity $2^{11}L-J^3$ positive,*

* Sylvester has inadvertently stated his condition to be that $2^{11}L-J^3$ is negative. It is easy to see, however, that what he has proved is, that this quantity must be positive. For the block which he has described lies on the side of the curve $2^{11}L-J^3$ next to the axis $L=0$. But when L is 0 and J negative, $2^{11}L-J^3$ is positive.

and L negative, which also infers J negative. If either condition fails, our roots are imaginary. It is supposed that in both cases D is positive.

241. We have seen that the cylinder parallel to the axis of z and standing on the curve $2^{11}L - J^3$ does not meet G above the plane D; the two values of z being one 0, the other negative. Any other surface then standing on the same curve and not meeting G would serve equally well as a wall of separation between the two classes of facultative points. For, all the points between the cylinder and this surface would be non-facultative, and therefore irrelevant to the question. Sylvester has thus seen that we may substitute for the criterion $2^{11}L - J^3$, $2^{11}L - J^3 + \mu JD$, provided that the second represent a surface not meeting G above the plane D. And on investigating within what limits μ must be taken, in order to fulfil this condition, he finds that μ may be any number between 1 and -2.

He avails himself of this to give criteria expressed as symmetrical functions of the roots. In the first place

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\delta-\varepsilon)^4$$

is an invariant (Art. 136), and being of the same order and weight as J can only differ from it by a numerical factor, which factor must be negative, since this function is essentially positive; and J we have seen is essentially negative when the roots are all real. And secondly, the symmetric function

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\varepsilon-\alpha)^4(\varepsilon-\beta)^4(\varepsilon-\gamma)^4(\delta-\alpha)^4(\delta-\beta)^4(\delta-\gamma)^4,$$

(the relation of which to the other may be seen by writing it in the form $D^2\Sigma(\alpha-\beta)^{-2}(\beta-\gamma)^{-2}(\gamma-\alpha)^{-2}(\delta-\varepsilon)^{-4}$, where D is the discriminant), is also an invariant, and of the twelfth order. It must therefore be of the form $\alpha J^3 + \beta JD + \gamma L$. Now, by using the quintic* $x(x^2-a^2)(x^2-b^2)$, the symmetric function

* It was observed, Art. 229, that the characteristic of this form is that Hermite's invariant I vanishes, hence it may be safely used in calculating any invariant function whose order is divisible by 4 and is below 36, since such forms cannot contain I, but though this form may be safely used in this case, it cannot always be safely used. For when a linear factor of a quintic is also a factor in the sextic covariant of the remaining quartic, a relation must exist between the invariants.

may easily be calculated and identified with the invariants; and the result is that its value is proportional to $2^{11}L - J^3 + \frac{4}{5}JD$. Since then the numerical multiplier of JD is within the prescribed limits, it may be used as a criterion, and Prof. Sylvester's result is, that the two symmetrical functions mentioned are such that not only are both positive, as is evident, if the roots are all real, but also if both are positive, and D positive, the roots *must* be all real. It ought to be possible to verify this directly by examining the form of these functions in the case of an equation with four imaginary roots.

242. I have also tried to verify these results by examining the invariants of the product of a linear factor and a quartic, $(\alpha x + \beta y)(x^4 + 6mx^2y^2 + y^4)$; these being necessarily covariants of the quartic (Art. 201). The coefficients of the quintic are then 5α, β, $3m\alpha$, $3m\beta$, α, 5β; and I find for the J of the quintic, $48\,(8SH - 3TU)$, or 48 times

$$(5m + 27m^3)(\alpha^4 + \beta^4) + (8 - 18m^2 - 54m^4)\alpha^2\beta^2.$$

Now the roots of the quartic are all real when m is negative, and when $9m^2$ is greater than 1. On inspection of the value given for J, we see that when m is negative every term but one is negative. Giving then m its smallest negative value $-\frac{1}{3}$, J is negative, viz. $-144(\alpha^2 - \beta^2)^2$; and J is *á fortiori* negative for every greater negative value of m. Or we may see the same thing by supposing $\beta = 0$, when we have only to look at the coefficient of the highest power of α in $8SH - 3TU$, which is $-8(b^2 - ac)S - 3Ta$. But now if we call the three Sturmian constants A, B, C, viz.

$$A = b^2 - ac,\quad B = 2SA + 3Ta,\quad C = S^3 - 27T^2,$$

the value given for J becomes $-6AS - B$, which is essentially negative when the roots are all real.

The invariant L, according to my calculation, is

$$54\,(8SH - 3TU)^3 - 6400\,(S^3 - 27T^2)(4H^3 - SHU^2 + TU^3)$$
$$+ 150\,(S^3 - 27T^2)U^2(8SH + 15TU) - 4050\,U^2S^3(2SH - 3TU),$$

whence $2^{11}L - J^3$ differs only by a positive constant multiplier from

$$-128\,(S^3 - 27T^2)(4H^3 - 3SHU^2 + TU^3)$$
$$+ 3\,(S^3 - 27T^2)U^2(8SH + 15TU) - 81\,U^2S^3(2SH - 3TU).$$

Writing $\theta = a^2d - 3abc + 2b^3$, the coefficient of the highest power of a in this is

$$128\,C\theta^2 + 81a^2S^3 + 45a^2CB - 54a^2USA.$$

All the terms of this but one are positive when the roots are all real, but as there is one negative term, is it not obvious, on the face of the formula, that the whole will be positive when the roots are all real. Still less that if this formula be positive and J negative, the roots are necessarily all real. Therefore, although no doubt Prof. Sylvester's rule may be tested by the process here indicated, to do so requires a closer examination of this formula than I am able to give.*

243. Prof. Cayley in his Eighth Memoir on Quantics (*Phil. Trans.* 1867), proceeded by a method a little different from that described above. Adopting as coordinates

$$x = \frac{2^{11}L - J^3}{J^3}, \quad y = \frac{D}{J^2}, \quad z = J,$$

then from the foregoing equation

$$16I^2 = JK^4 + 8LK^3 - 2J^2LK^2 - 72JL^2K - 432L^3 + J^3L^2,$$

where, $K = \frac{1}{128}(J^2 - D)$, we deduce without much difficulty

$$2.2^{32}\frac{I^2}{J^9} = -3x^3 - x^2 + y(72x^2 + 205x + 125) + y^2(-29x - 17)$$
$$+ y^3(-x - 9) + y^4.2 = \phi(x, y) \text{ suppose};$$

or, since $z = J$, we have $z\phi(x, y) = 2.2^{32}\frac{I^2}{J^8} = \text{positive}$,

and the surface $G = 0$ may be replaced by $z\phi(x, y) = 0$; that is, by the plane $z = 0$ and the cylinder $\phi(x, y) = 0$. The configuration of the regions into which space is divided by this surface depends only on the form of the curve $\phi(x, y) = 0$ (Prof. Sylvester's "Bicorn"), which is the section of the cylinder by the plane $z = 0$, and the discussion as to the reality of the roots may be then effected by means of the plane curve alone; the results, of course, agree with those obtained above.

* The verification, however, is easy in the particular case $x(x^4 + 6mx^2y^2 + y^4)$. We have then $J = 48m(5 + 27m^2)$, $L = 12m(5 - 9m^2)^4$; $2^{11}L - J^3$ proportional to $m(1 - 9m^2)(50 + 45m^2 + 648m^4 + 729m^6)$. Thus, when m is negative, and $9m^2 > 1$, we have J and L negative and $2^{11}L - J^3$ positive. The latter is positive for imaginary roots only when m is positive, but in this case J is positive. The imaginary roots must, therefore, be detected by one criterion or other.

244. It has been already mentioned (Art. 231), that M. Hermite has made use of the fact, that the quintic as well as every equation of odd degree is reducible to a *forme-type*, in which the x and y are linear covariants and the coefficients are invariants. It follows immediately, that by applying Sturm's theorem to the *forme-type*, the conditions for reality of roots may be expressed by invariants.

Hermite extends his theorem to equations of even degree above the fourth, by the method indicated in Art. 248. Writing $J^2 - 3K = M$, $JK + 9L = N$; and Q a numerical multiple of Hermite's I, such that

$$Q^2 = JK^2M^2 - 2MNK(J^2 + 12K) + JN^2(J^2 + 72K) - 48N^3,$$

then the coefficients of the *forme-type* are

$$A = QM,$$

$$B = JKM^2 - MN(J^2 + 18K) + 30JN^2,$$

$$C = Q(JM - 12N),$$

$$D = J^2KM^2 - JMN(J^2 + 30K) + N^2(42J^2 + 144K),$$

$$E = Q(J^2M - 24JN),$$

$$F = J^3KM^2 - J^2MN(J^2 + 42K) + N^2J(54J^2 + 288K) - 1152N^3.$$

Thus the first Sturmian constant $B^2 - AC$ is found to be

$$36N^2\{(MK - 5JN)^2 - 16MN^2\}.$$

The Sturmian constants being essentially unsymmetrical, there seems no reason to expect that the discussion of these forms would lead to any results of practical interest. The coefficients of the *forme-type*, as M. Hermite remarked, satisfy the relations

$$AJ^2 - 2CJ + E = 0, \quad BJ^2 - 2DJ + F = -1152N^3,$$

$$AE - 4BD + 3C^2 = -12^4N^5, \quad AF - 3BE + 2CD = 0,$$

$$BF - 4CE + 3D^2 = 12^4JN^5.$$

Thus then the quadratic covariant is $N^5(x^2 - Jy^2)$; and operating with this on the quintic, we get the canonizant in the form

$$N^5(AJ - C,\ BJ - D,\ CJ - E,\ DJ - F)(x, y)^3;$$

the coefficients inside the parentheses being all further divisible by N. Hence we have

$$ACE + 2BCD - AD^2 - EB^2 - C^3 = -4.12^4N^6Q,$$

and the second Sturmian constant is got immediately by substituting in the formula of Art. 235, the values just found for

$$B^2 - AC,\ AE - 4BD + 3C^2,\ ACE + 2BCD - \text{\&c.},^*$$

245. The Tschirnhausen transformation consists in taking a new variable

$$y = \alpha + \beta x + \gamma x^2 + \ldots + \lambda x^{n-1};$$

then there are n values of y corresponding to the n values of x, and the coefficients of the new equation in y are readily found in terms of those of the given equation by the method of symmetric functions, the first for example being $\alpha s_0 + \beta s_1 + \gamma s_2 + \text{\&c.}$ The coefficient of y^{n-1} is evidently a linear homogeneous function of α, β, &c., that of y^{n-2} a quadratic, of y^{n-3} a cubic function, and so on. In the case of the quintic, the transformation is $y = \alpha + \beta x + \gamma x^2 + \delta x^3$, and we have four constants α, β, γ, δ at our disposal. Mr. Jerrard pointed out that the coefficient of y^3 being a quadratic function of α, β, γ, δ was (Art. 165) capable of being written as the algebraic sum of four squares, say $t^2 - u^2 + v^2 - w^2$. It can therefore be made to vanish, by assuming two linear relations between α, β, γ, δ; $t - u = 0$, $v - w = 0$. If we combine with these two that linear relation which makes the coefficient of y^4 vanish, we have three relations enabling us to express three of the constants α, β, γ, δ linearly in terms of the fourth. We can then by solving a cubic make the coefficient of y^2 also vanish, or else by solving a biquadratic make the coefficient of y vanish. In this way Mr. Jerrard showed, that by the solution of equations of inferior orders, a quintic may be reduced to either of the trinomial forms $y^5 + by = c$, or $y^5 + by^2 = c$. The actual performance of the

* The coefficients of the *forme-type* of the quintic were given by M. Hermite (*Cambridge and Dublin Mathematical Journal*, 1854, vol. IX. p. 193), and re-calculated by me before I found out the key for the translation of Hermite's notation into Sylvester's, which is $\Delta = J$, $J_2 = -K$, $J_3 = JK + 9L$.

The discussion of the invariantive characteristics of the reality of the roots of a quintic was originally commenced by M. Hermite in the same classical paper, and was resumed by him in his valuable memoir presented to the French Academy, t. 62, 1866. His result, in our notation, is that the roots are all real, when the discriminant being positive, we have also positive K, $2^{11}L - J^3 + JD$, and $K(JL + K^2) - 18L^2$. It seems to me that this result is superseded by the greater simplicity of Prof. Sylvester's criteria.

transformations would be a work of great labour, but M. Hermite showed how, by somewhat altering the form of substitution, we can avail ourselves of the help of invariants.

If we have to transform the equation $ax^n + bx^{n-1} + cx^{n-2} + \&c.$, Hermite assumes

$$y = a\lambda + (ax + b)\,\alpha + (ax^2 + bx + c)\,\beta + (ax^3 + bx^2 + cx + d)\,\gamma + \&c\,,$$

then in the first place the transformed equation will be divisible by α; and secondly, if the given equation be linearly transformed, and if the corresponding substitution for the transformed equation be

$$Y = A\lambda' + (AX + B)\,\alpha' + (AX^2 + BX + C)\,\beta' + \&c.,$$

then he has shewn that the expressions for α', β', &c. in terms of α, β, &c. involve only the coefficients of linear transformation, and not those of the given equation. It is not so with respect to the first coefficient λ, which we have therefore designated by a special letter. But the theory of linear substitutions will be directly applicable to all functions of the coefficients of the transformed equation which do not contain λ. Such, for example, will be all symmetric functions of the differences of the roots of the new equation, since, on subtracting

$$y_1 = a\lambda + (ax_1 + b)\,\alpha + \&c.,\quad y_2 = a\lambda + (ax_2 + b)\,\alpha + \&c.,$$

λ disappears. Or, what comes to the same thing, if we take λ such that the coefficient of y^{n-1} in the new equation shall vanish, then the theory of linear substitutions is applicable to all the coefficients of the transformed. I give Cayley's proof of Hermite's theorem, and, after his example, take, to fix the ideas, the quartic

$$(a, b, c, d, e\chi x, 1)^4.$$

Then, as we have used binomial coefficients, we shall write the equation of transformation

$$y = a\lambda + (ax + 4b)\,\gamma - (ax^2 + 4bx + 6c)\,\beta + (ax^3 + 4bx^2 + 6cx + 4d)\,\alpha.$$

Adding the 4 values of y, and employing Newton's formulæ for the sums of powers of the roots, we see that the coefficient of y^{n-1} in the transformed equation will vanish if

$$a\lambda + 3b\gamma - 3c\beta + d\alpha = 0.$$

This reduces the value of y to

$$(ax+b)\,\gamma-(ax^2+4bx+3c)\,\beta+(ax^3+4bx^2+6cx+3d)\,\alpha.$$

In general it will be observed, that in this substitution all the terms have the binomial coefficients corresponding to the order of the given equation, except the terms not involving x, which have the binomial coefficients answering to the order one lower.

246. Now what is asserted is, that all the coefficients of the transformed equation will be invariants of the system

$$(a, b, c, d, e\,\text{][}\,x, y)^4,\quad (\alpha, \beta, \gamma\,\text{][}\,x, y)^2,$$

and of course if we regard y as constant, the whole transformed function will be such an invariant.

This will be proved by shewing that it is made to vanish by either of the operations

$$a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+4d\frac{d}{de}+\left(\alpha\frac{d}{d\beta}+2\beta\frac{d}{d\gamma}\right),$$

$$4b\frac{d}{da}+3c\frac{d}{db}+2d\frac{d}{dc}+e\frac{d}{dd}+\left(2\beta\frac{d}{d\alpha}+\gamma\frac{d}{d\beta}\right).$$

Let the general substitution be $y=V$, and let V_1, V_2, &c. be what V becomes when we substitute for x each of the roots of the given equation, the transformed in y is the product of the factors $y-V_1$, $y-V_2$, &c., and it is sufficient to prove that each of these factors is reduced to zero by this differentiation. We may, as in Art. 64, write the first part of the first operation $\frac{d}{d\zeta}$, and in order to calculate $\frac{dV}{d\zeta}$, we must find $\frac{dx}{d\zeta}$. Operating on the given equation, we get

$$(a, b, c, d\,\text{][}\,x, 1)^3\frac{dx}{d\zeta}+(a, b, c, d\,\text{][}\,x, 1)^3=0,\text{ or }\frac{dx}{d\zeta}=-1.$$

The part then of the differential of V which depends on the variation of x is

$$-\{a\gamma-(2ax+4b)\,\beta+(3ax^2+8bx+6c)\,\alpha\},$$

and the part got by directly operating on the a, b, &c. which explicitly appear in V is

$$a\gamma-(4ax+6b)\,\beta+(4ax^2+12bx+9c)\,\alpha.$$

OF THE UNIVERSITY OF CALIFORNIA

Adding, we have

$$\frac{dV}{d\zeta} = -2(ax+b)\beta + (ax^2+4bx+3c)\alpha = -\left(\alpha\frac{dV}{d\beta} + 2\beta\frac{dV}{d\gamma}\right),$$

which proves that the effect of the first operation on V is zero.

In like manner, for the second operation, we have, by performing on the original equation,

$$(a, b, c, d\,(x, 1)^3\frac{dx}{d\eta} + x(b, c, d, e\,(x, 1)^3 = 0.$$

But the original equation may be written

$$x(a, b, c, d\,(x, 1)^3 + (b, c, d, e\,(x, 1)^3 = 0.$$

Hence $\frac{dx}{d\eta} = x^2$. The part of $\frac{dV}{d\eta}$ due to the variation of x is therefore

$$ax^2\gamma - (2ax^3 + 4bx^2)\beta + (3ax^4 + 8bx^3 + 6cx^2)\alpha.$$

The remaining part is

$$(4bx+3c)\gamma - (4bx^2+12cx+6d)\beta + (4bx^3+12cx^2+12dx+3e)\alpha.$$

Adding, the coefficient of α vanishes in virtue of the original equation, and the remaining part is found to be

$$-\left(\gamma\frac{dV}{d\beta} + 2\beta\frac{dV}{d\alpha}\right),$$

which completes the proof of the theorem.

247. When this transformation is applied to a cubic, if we consider α, β as variables, the coefficients of the transformed equation in y will be covariants of the given equation. The transformed in fact has been calculated by Prof. Cayley, and found to be $y^3+3Hy+J$, where H is the Hessian $(ac-b^2)\alpha^2+$ &c., and J is the covariant (Art. 142), $(a^2d - 3abc + 2b^3)\alpha^3 +$ &c.

Prof. Cayley has also calculated the result of transformation as applied to a quartic. Take the two quantics, as in Art. 212,

$$(a, b, c, d, e\,(x, y)^4,\ (\alpha, \beta, \gamma\,(x, y)^2,$$

and let C denote the skew invariant of the same article, p. 203; let S and T denote the two invariants of the quartic; also let $\Sigma' = 6\Delta$, then the transformed function in y is

$$y^4 + 6(\Sigma + S\Delta)y^2 + 4Cy + S\phi^2 - 3\Sigma^2 - 6S\Sigma\Delta + 18T\phi\Delta.$$

Prof. Cayley has also calculated the S and T of the transformed equation. In making the calculation, it is useful to observe that since the square of J, from which C was derived (p. 204), can be expressed in terms of the other invariants, so also may the square of C; the actual expression derived from his being in our notation

$$-C^2 = T\phi^3 - S\Sigma\phi^2 - 9(2\Sigma + S\Delta)T\Delta\phi + (2\Sigma + 3S\Delta)^2\Sigma + 54T^2\Delta^3.$$

The result then is that the new S is $S\phi^2 + 3S^2\Delta^2 + 18T\Delta\phi$, and the new T is $T\phi^3 + S^2\Delta\phi^2 + 9\Delta^2 ST\phi + \Delta^3(54T^2 - S^3)$.

Finally, he has observed that these are the S and T of $U\phi + 6H\Delta$, as may be verified by the formulæ of Art. 210. It follows, then, that the effect of the Tschirnhausen transformation is always to change a quartic into an equation having the same invariants as one of the form $U + \lambda H$, and, therefore, reducible by linear transformation to the latter form. The foregoing results in a different notation are reproduced, and the corresponding results for the quintic are obtained in Prof. Cayley's Memoir on Tschirnhausen's Transformation, *Phil. Trans.*, vol. CLII. (1862).

248. The following is the form in which M. Hermite presented his theory, and applied it to the case of the quintic.

Let u be a quantic $(x, y)^n$; u_1, u_2 its differentials with regard to x and y; let ϕ be a covariant, which we take of the degree $n-2$ in order that the equation we are about to use may be homogeneous in x and y; then the coefficients of the transformed equation, obtained by putting $z = \frac{y\phi}{u_1}$, are all invariants of u. The equation in z is got by eliminating x and y between $zu_1 - y\phi = 0$, and $u = 0$, or, what comes to the same thing, $zu_2 + x\phi = 0$, which follows from the other two. If we linearly transform x and y, the new equation in z is got, in like manner, by eliminating between $zU_1 - Y\Phi = 0$, $zU_2 + X\Phi = 0$. But, if $x = \lambda X + \mu Y$, $y = \lambda' X + \mu' Y$, $\Delta = \lambda\mu' - \lambda'\mu$, we have

$$\Delta X = \mu' x - \mu y, \quad \Delta Y = \lambda y - \lambda' x,$$

and Art. 130, $U_1 = \lambda u_1 + \lambda' u_2$, $U_2 = \mu u_1 + \mu' u_2$, and, since ϕ is a covariant, we have $\Phi = \Delta^i\phi$. Making these substitutions, the equation in z, corresponding to the transformed equation, is got

by eliminating between

$$z(\lambda u_1 + \lambda' u_2) - \Delta^{t-1}\phi(\lambda y - \lambda' x) = 0,$$
$$z(\mu u_1 + \mu' u_2) + \Delta^{t-1}\phi(\mu' x - \mu y) = 0.$$

Multiply the first by μ', the second by λ', and subtract, and we have $\Delta z u_1 - \Delta^t y\phi = 0$. In like manner, multiplying the first by μ, the second by λ, and subtracting, we get $\Delta z u_2 + \Delta^t x\phi = 0$. In other words, we have the two original equations, except that z is divided by Δ^{t-1}. Consequently, the equations in z corresponding to the original equation, and to the same linearly transformed, only differ in having the powers of z multiplied by different powers of the modulus of transformation Δ, and therefore the several coefficients of the powers of z are invariants.

The actual form of the equation in z will be

$$z^n + \frac{A}{D} z^{n-2} + \frac{B}{D} z^{n-3} + \&c. = 0.$$

It is easy to see that the discriminant will appear in the denominator; and the coefficient of z^{n-1} will vanish, since, if ϕ be any function of the order $n-2$, the sum of the results of substituting all the roots of U in $\frac{\phi}{U_1}$ vanishes. In fact, when the terms of this sum are brought to a common denominator, the numerator is the sum of $\phi\alpha$ multiplied by the differences of all the roots except α, and this is a function of the order $n-2$ in α, which vanishes for $n-1$ values of α, $\alpha = \beta$, $\alpha = \gamma$, &c., and must therefore be identically nothing.

In applying this method to the quintic $(x, 1)^5$, Hermite substitutes

$$zU_1 = \alpha\phi_1 + \beta\phi_2 + \gamma\phi_3 + \delta\phi_4,$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are four covariant cubics of the orders 3, 5, 7, 9 respectively in the coefficients. ϕ_1 is the canonizant. ϕ_2 is the covariant cubic of the fifth order, the Jacobian of S and T whose leading term or *source*, whence all the other terms can be derived, is printed in full as (11) Art. 232; on inspection we see that this source vanishes if both a and b vanish; consequently, if the given quintic has two equal roots, their

common value satisfies this covariant. We can form a covariant cubic of the seventh order from ϕ_2 in the same way that ϕ_2 was formed from ϕ_1, and by adding ϕ_1, multiplied by J and a numerical coefficient, can obtain ϕ_3, such that its source vanishes when a and b vanish; and, in like manner ϕ_4 can be made to possess the same property.

When this substitution is made, the coefficient of z^3 is a quadratic function of α, β, γ, δ. Hermite finds for its actual value (a result which may be verified by working with the special form, note, p. 248),

$$\{F\alpha^2 + 6KD\alpha\gamma - D(F + 10JK)\gamma^2\} + D\{K\beta^2 + 2F\beta\delta - (9KD + 10AF)\delta^2\},$$

where $F = 9(16L - JK)$, which vanishes when the quintic has two distinct pairs of equal roots. By breaking up into factors each of the parts into which this coefficient has been divided, the two linear relations between α, γ; β, δ, which will make it to vanish, can readily be obtained; as also by another process which I shall not delay to explain. The discussion of this coefficient is also the basis of Hermite's later investigations as to the criteria for reality of the roots. He avails himself of a principle of Jacobi's (*Crelle*, vol. L.), that if α, β, γ, &c. be the roots of a given equation, and if the quadratic function

$$(t + \alpha u + \alpha^2 v + \ldots \alpha^{n-1} w)^2 + (t + \beta u + \beta^2 v + \&c.)^2 + \&c.,$$

be brought by real substitution to a sum of squares, the number of negative squares will be equal to the number of pairs of imaginary roots in the equation. Hermite shews, by an easy extension of this principle, that the number of pairs of imaginary roots of the quintic is found by ascertaining the number of negative squares, when the coefficient of z^3 just written is resolved into a sum of squares. And since the same process is applicable to every equation whose degree is above the fourth, he concludes that the conditions for reality of roots in every equation above the fourth can be expressed by invariants.

249. It does not enter into the plan of these Lessons to give an account of the researches to which the problem of

resolving the quintic has given rise.* The following, however, finds a place here on account of its connection with the theory of invariants. Lagrange, as is well known, made the solution of a quintic to depend on the solution of a sextic; and it can easily be proved that functions of five letters can be formed capable of six values by transposition of letters. Let 12345 denote any cyclic function of the roots of a quintic; such, for example, as the product

$$(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\alpha)^2,$$

where evidently 23451 and 15432 would denote the same as 12345; then it can easily be seen that there can be written down in all twelve such cyclic functions. But, further, these distribute themselves into pairs; and by so grouping them we can form a function capable of only six values; for instance, 12345 + 13524, 12435 + 14523, 13245 + 12534, 13425 + 14532, 14235 + 12543, 14325 + 13542. The actual formation of the sextic having these values for its roots is in most cases a work of extreme labour. M. Hermite, however, pointed out that when the function 12345 is the product of the squares of differences written above,† all the coefficients of the corresponding sextic are invariants, and that the calculation therefore is practicable. I have thought it desirable actually to form the equation, because, in the development of the theory of sextics, it will be necessary to ascertain the invariant characteristics of sextics whose solution depends on that of a quintic; and it may be useful to be in possession of more than one of the sextics which spring out of the discussion of a quintic.‡ I take the simple example $x^5+2mx^3y^2+xy^4$, of which,

* Among the most remarkable of recent investigations in this subject is the application to it of the theory of elliptic functions by M. Hermite and M. Kronecker.

† In the method of Messrs. Harley and Cockle, the function 12345 is

$$\alpha\beta+\beta\gamma+\gamma\delta+\delta\varepsilon+\varepsilon\alpha,$$

and the sextic chosen is that whose roots are 12345 − 13524, &c. This has been calculated by Prof. Cayley (*Philosophical Transactions*, 1861, p. 263), and the result is very simple, two terms of the sextic are wanting; but the coefficients are not invariants.

‡ The form arrived at by M. Kronecker and M. Brioschi is

$$(x-a)^5(x-5a)+10b(x-a)^3-c(x-a)+5b^2-ac=0.$$

By the help of the formulæ given further on, the invariants of this equation can be calculated, and a, b, c eliminated.

since two pairs of roots are equal with opposite signs, the functions of the differences can easily be formed. I find then that the sextic is the product of

$$t^2 + 2^6 (m + m^3)\, t + 2^{10} (m^6 - 2m^4 + 5m^2),$$

by the square of

$$t^2 + 2^4 (m^3 + 3m)\, t + 2^6 (m^6 + 5m^4 + 19m^2 - 25).$$

But if we first multiply the quintic by five, its invariants are

$$J = 2^4 m (5 + 3m^2),\ D = 2^8 . 5^3 (1 - m^2)^2,\ L = 4m (5 - m^2)^4.$$

To avoid fractions I write $J = 2A$, $D = 250B$, $J^3 - 2^{11} L = 50C$; and then forming the sextic, and expressing its coefficients in terms of the invariants, I obtain

$$t^6 + 4At^5 + (6A^2 - 25B)\, t^4 + (4A^3 + 2C - 30AB)\, t^3$$
$$+ t^2 (A^4 + 4AC - 17A^2B + \tfrac{625}{4} B^2)$$
$$+ t (2A^2C - 4A^3B - 7BC + 110AB^2) + C^2 - 4ABC + 20A^2B^2,$$

which is a perfect square, as it ought to be, when $D = 0$.*

250. M. Hermite has studied in detail the expression of the invariants in terms of the roots. He uses the equation transformed so as to want the first and last terms; that is to say, so that one root is 0 and another infinite; and the calculation is thus reduced to forming symmetric functions of the roots of a cubic. I had been led independently to try the same transformation on the problem discussed in the last Article, but found that, even when thus simplified, the problem remained a difficult one. It would be necessary to form for a cubic the sextic whose roots are the six values of

$$\alpha^2 (\beta - \gamma)^2 (\alpha - \gamma)^2 + \beta^2 \gamma^2 (\alpha - \beta)^2,$$

and then to identify the result with combinations of the forms assumed by the invariants of the quintic when a and f vanish. M. Hermite expresses his own invariant I as follows. Let

$$\alpha_1 = (\alpha - \beta)(\alpha - \varepsilon)(\delta - \gamma) + (\alpha - \gamma)(\alpha - \delta)(\beta - \varepsilon),$$
$$\alpha_2 = (\alpha - \beta)(\alpha - \gamma)(\varepsilon - \delta) + (\alpha - \delta)(\alpha - \varepsilon)(\beta - \gamma),$$
$$\alpha_3 = (\alpha - \beta)(\alpha - \delta)(\varepsilon - \gamma) + (\alpha - \gamma)(\alpha - \varepsilon)(\delta - \beta);$$

the continued product of these is symmetrical with respect to all the roots except α; and if we multiply this product by the similar products obtained for the other four roots we get I.

* Though the form with which I have worked is a special one, I believe that the result is general; because it seemed to me that the coefficients only admitted of being expressed in terms of the invariants in one way.

These factors are of course the values of the determinants

$$\begin{vmatrix} \alpha^2, & 2\alpha, & 1 \\ \beta\delta, & \beta+\delta, & 1 \\ \gamma\varepsilon, & \gamma+\varepsilon, & 1 \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} 2\alpha-(\gamma+\varepsilon), & \alpha(\gamma+\varepsilon)-2\gamma\varepsilon \\ 2\alpha-(\beta+\delta), & \alpha(\beta+\delta)-2\beta\delta \end{vmatrix}, \text{ &c.,}$$

which express, p. 193, that one of the roots is self-conjugate of the involution determined by the other four, which is the case when I vanishes, as remarked, Art. 229. Determining the numerical constant by a special form, such as $ax^5+5exy^4+fy^5=0$, we find the product of these fifteen factors by a^{18} to be $10^{15}I$.

251. From the roots of a quintic five sets of four can be formed by omitting each in turn, let S_α, S_β, S_γ, S_δ, S_ε denote the equi-anharmonic functions of these sets of four, see Art. 199; also T_α, T_β, T_γ, T_δ, T_ε their harmonic functions; it is easy to see by comparing terms in a simple case, for instance, for the quintic $ax^5+10cx^3y^2=0$, that we have in terms of the roots

$$\frac{200}{a^2}S = S_\alpha(x-\alpha y)^2+S_\beta(x-\beta y)^2+S_\gamma(x-\gamma y)^2+S_\delta(x-\delta y)^2+S_\varepsilon(x-\varepsilon y)^2$$

$$-\frac{6000}{a^3}T = T_\alpha(x-\alpha y)^3+T_\beta(x-\beta y)^3+T_\gamma(x-\gamma y)^3+T_\delta(x-\delta y)^3+T_\varepsilon(x-\varepsilon y)^3$$

$$-100H = a^2\Sigma(\alpha-\beta)^2(x-\gamma y)^2(x-\delta y)^2(x-\varepsilon y)^2.$$

If we calculate for the quintic $ax^5+10dx^2y^3=0$, the value of the ten terms $\Sigma(\alpha-\beta)^4(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\beta)^2$ which we saw, Art. 241, can only be the quartinvariant, we find

$$a^4\Sigma(\alpha-\beta)^4(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\gamma)^2 = -1250J.$$

The function $\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\alpha)^2$ containing twelve terms (Art. 248), is found to have precisely the same value.

Similarly it may be noticed, for the covariant D, note p. 240,

$$a^4\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\gamma)^2(x-\alpha y)^2(x-\beta y)^2 = 1000D,$$

whose source is the function D written in full on page 230.

The value of the discriminant R on the same page must be multiplied by 5^5 to become identical with a^8 times the product of the squares of the differences.

The condition that four of the five roots may be equi-anharmonic will be that some one of S_α, S_β, &c. vanish. Hence their product will be an invariant. We get by a special form

$$a^8S_\alpha S_\beta S_\gamma S_\delta S_\varepsilon = 20^5(J^2-3K).$$

Similarly for the harmonic condition, we get the invariant

$$a^{12}T_\alpha T_\beta T_\gamma T_\delta T_\varepsilon = 5^{10}(2^{13}3^3L + 2^7 3^2 JK - J^3).$$

Again, the function $\Sigma(\alpha-\beta)^2(\alpha-\gamma)^2(\delta-\varepsilon)^2$ consists of six terms belonging specially to α, which we may denote by H_α, and similarly for H_β, &c. It is easy to see that, if we write α for $x : y$ in the above value for the covariant S, the right side becomes $2H_\alpha$. Hence, the eliminant of S and u is the product of these five factors. But by Sylvester's canonical form or otherwise, that eliminant is found to be $2JK - 9L - J^3$, and by a special quintic such as $(x-1)^2(x+2)x^2 = 0$, we can find the value of the constant, thus

$$3a^{12}H_\alpha H_\beta H_\gamma H_\delta H_\varepsilon = 10^2(2JK - J^3 - 9L).$$

Similarly we can see that I is the eliminant of T and u, for, the result of substituting α in the above value of T is six times the product of the factors α_1, α_2, α_3.

Again, by taking a special quintic of the form used in Art. 241, we find the constant which gives the symmetric function,

$$a^{12}\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\varepsilon-\alpha)^4(\varepsilon-\beta)^4(\varepsilon-\gamma)^4(\delta-\alpha)^4(\delta-\beta)^4(\delta-\gamma)^4$$
$$= \tfrac{1}{2}.5^{10}(2^{11}L - J^3 + \tfrac{1}{6}JD) = 2.5^9(5.2^9L - J^3 - 2^5JK).*$$

252. *The Sextic.* The investigations of Clebsch and Gordan show that, including the sextic itself, there are in all 26 forms. There are four independent invariants, which we shall call A, B, C, D, of the orders 2, 4, 6, 10 respectively; a fifth E, of the order 15, is skew and its square a rational and integral function of the other four. There are six quadric covariants whose orders in the coefficients are respectively 3, 5, 7, 8, 10, 12; five quartics of orders 2, 4, 5, 7, 9; five sextics, orders 1, 3, 4 and two of the sixth; three octavics, orders 2, 3, 5; one decimic, order 4; and one duodecimic of order 3.

The first invariant A is $\overline{12^6}$, formed by the method of Art. 141, and for the general sextic is

$$ag - 6bf + 15ce - 10d^2.$$

I have given (Art. 174) the canonical form of the sextic; but I believe it will be found in practice not less convenient to use the more general form

$$au^6 + bv^6 + cw^6 + dz^6.$$

* The equation determining the anharmonic ratios of the roots has been given by Mr. M. J. M. Hill. *Proc. Lond. Math. Soc.*, vol. XIV.

To this we should be led by the theory of two quintics, which cannot be more simply expressed than as each the sum of four fifth powers. For the form just given, the invariant A is, by proceeding as in Art. 223, found to be $\Sigma ab\,(12)^6$, or, in full,

$$ab\,(12)^6 + ac\,(13)^6 + ad\,(14)^6 + bc\,(23)^6 + bd\,(24)^6 + cd\,(34)^6.$$

The Hessian of the sextic, $\overline{12^2}$, is of the eighth degree, the general coefficients being $ac - b^2$, $4\,(ad - bc)$, $6ae + 4bd - 10c^2$, $4af + 16be - 20cd$, $ag + 14bf + 5ce - 20d^2$,* &c.; and for my canonical form is $\Sigma abu^4v^4\,(12)^2$. The sextic has another covariant of the second order in the coefficients, with the variables in the fourth degree, viz. the S of the emanant quartic, which is for the canonical form $\Sigma abu^2v^2\,(12)^4$, the general coefficients being

$$ae - 4bd + 3c^2,\quad 2af - 6be + 4cd,\quad ag - 9ce + 8d^2*,\ \&c.$$

To these we add, the covariant sextic, of the third order, the T of the quartic emanant, which for the canonical form is $\Sigma abc\,(12)^2\,(23)^2\,(31)^2\,u^2v^2w^2$, and whose general coefficients are

$$ace + 2bcd - ad^2 - eb^2 - c^3,\quad 2acf - 2ade - 2b^2f + 2bce + 2bd^2 - 2c^2d,$$
$$acg + 2adf - 3ae^2 - b^2g - 2bcf + 4bde + 2c^2e - 3cd^2,$$
$$2adg - 2aef - 2bcg + 4bdf - 2be^2 - 2c^2f + 6cde - 4d^3*,\ \&c.$$

Also, the simplest quadricovariant l, of the third order, $\overline{12^4}$ of S and u, or $\overline{12^6}$ of u and H, which for the canonical form is $\Sigma abc\,(23)^4\,(31)^2\,(12)^2\,u^2$, and whose general coefficients are

$$acg - b^2g - 3adf + 3bcf + 2ae^2 - bde - 3c^2e + 2cd^2 = l_0,$$
$$adg - bcg - aef - 8bdf + 9c^2f + 9be^2 - 17cde + 8d^3 = 2l_1,$$
$$aeg - 3bdg + 2c^2g - af^2 + 3bef - cdf - 3ce^2 + 2d^2e = l_2.$$

253. We take for the invariant B that which has been called by Sylvester the *catalecticant*, which expresses the condition that the sextic should be reducible to the sum of three sixth powers, and is (Art. 171) the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}.$$

* It seems unnecessary to write the terms which follow from symmetry.

This expanded is

$$aceg - acf^2 - ad^2g + 2adef - ae^3 - b^2eg + b^2f^2 + 2bcdg - 2bcef - 2bd^2f + 2bde^2 - c^3g + 2c^2df + c^2e^2 - 3cd^2e + d^4.$$

If now we form the quadrinvariant of the Hessian, we find it proportional to $A^2 + 300B$; if that of the covariant S, we find $A^2 - 36B$; and if we operate on the sextic with the covariant T, we get B. Applying this last process then to the canonical form, we get the value of B,

$$abcd\,(12)^2\,(23)^2\,(34)^2\,(41)^2\,(13)^2\,(24)^2,$$

which vanishes, as it ought, if any of the quantities a, b, c, d vanishes, or if any two of the four functions u, v, w, z become identical.

254. We take for the form of the fundamental sextinvariant C, that which involves no power higher than the second of the leading coefficient a, and which for the general form is $C =$

$$\begin{aligned} &a^2d^2g^2 - 6a^2defg + 4a^2df^3 + 4a^2e^3g - 3a^2e^2f^2 - 6abcdg^2 + 18abcefg \\ &- 12abcf^3 + 12abd^2fg - 18abde^2g + 6abe^3f + 4ac^3g^2 - 24ac^2e^2g \\ &- 18ac^2dfg + 30ac^2ef^2 + 54acd^2eg - 12acd^2f^2 - 42acde^2f \\ &+ 12ace^4 - 20ad^4g + 24ad^3ef - 8ad^2e^3 + 4b^3dg^2 - 12b^3efg \\ &+ 8b^3f^3 - 3b^2c^2g^2 + 30b^2ce^2g - 24b^2cef^2 - 12b^2d^2eg - 24b^2d^2f^2 \\ &+ 60b^2de^2f - 27b^2e^4 + 6bc^3fg - 42bc^2deg + 60bc^2df^2 - 30bc^2e^2f \\ &+ 24bcd^3g - 84bcd^2ef + 66bcde^3 + 24bd^4f - 24bd^3e^2 + 12c^4eg \\ &- 27c^4f^2 - 8c^3d^2g + 66c^3def - 8c^3e^3 - 24c^2d^3f - 39c^2d^2e^2 + 36cd^4e - 8d^6. \end{aligned}$$

In terms of these we can express the other invariants of the sixth order. Thus, the cubinvariant of the covariant quartic is $A^3 - 108AB - 54C$; the cubinvariant of the Hessian, p. 141, is $3A^3 - 100AB + 2750C$; the discriminant of the quadratic l is $4(l_0l_2 - l_1^2) = 4AB + 3C$; and the quadrinvariant of the sextic covariant is $2AB - C$. The last-named invariant can be easily calculated in the case of the canonical form. We have to operate with $\Sigma abc\,(12)^2\,(23)^2\,(31)^2u^2v^2w^2$ on itself. Now if we operate with $u^2v^2z^2$ on $u^2v^2w^2$ the result is proportional to $(12)^2MN$, where M and N have the same meaning as in Art. 223; and if with $u^2v^2w^2$ on itself the result is $-(12)^2\,(23)^2\,(31)^2$. Hence we get for the invariant in question

$$\Sigma a^2b^2c^2\,(12)^6\,(23)^6\,(31)^6 - 2abcd\Sigma ab\,(12)^6M^3N^3.$$

255. If a, b, c all vanish, the invariants A, B, C become respectively $-10d^2$, d^4, $-8d^6$. Hence, when the sextic has as factor a perfect cube, the conditions must be fulfilled $A^2=100B$, $4AB=5C$, $AC=80B^2$. If we make a, b, f, g all $=0$, the invariants become

$$15ce-10d^2,\quad c^2e^2-3cd^2e+d^4,\quad -8c^3e^3-39c^2d^2e^2+36cd^4e-8d^6;$$

consequently when the sextic has two square factors, in addition to the discriminant vanishing, the condition must be satisfied,

$$(A^3-300AB+250C)^2=5\,(A^2-100B)^3.$$

256. If we make b, d, $f=0$ in the equation, the discriminant will be ag multiplied by the square of the discriminant of $(a, 5c, 5e, g\chi x, y)^3$; and if all the terms vanish but a, d, g, the discriminant will be a^2g^2 multiplied by the cube of the discriminant of $(a, 10d, g\chi x, y)^2$. Knowing these terms in the discriminant, the rest can be calculated by means of the differential equation. The resulting value of Δ is

	a^5g^5	$+43500$	$a^3bdef^2g^2$	$+30000$	$a^3d^4g^3$
-30	a^4bfg^4	-7500	a^3bdf^4g	-330000	$a^3d^3efg^2$
-300	a^4ceg^4	$+57000$	$a^3be^3fg^2$	$+50000$	$a^3d^3f^3g$
$+375$	$a^4cf^2g^3$	-97500	$a^3be^2f^3g$	$+250000$	$a^3d^2e^3g^2$
-300	$a^4d^2g^4$	$+37500$	a^3bef^5	$+675000$	$a^3d^2e^2f^2g$
$+3000$	a^4defg^3	$+1000$	$a^3c^3g^4$	-375000	$a^3d^2ef^4$
-2500	$a^4df^3g^2$	-27000	$a^3c^2dfg^3$	-900000	a^3de^4fg
$+1000$	$a^4e^3g^3$	$+18750$	$a^3c^2e^2g^3$	$+500000$	$a^3de^3f^3$
-7500	$a^4e^2f^2g^2$	$+16875$	$a^3c^2ef^2g^2$	$+250000$	a^3e^6g
$+9375$	a^4ef^4g	-9375	$a^3c^2f^4g$	-150000	$a^3e^5f^2$
-3125	a^4f^6	-7500	$a^3cd^2eg^3$	-2500	$a^2b^3dg^4$
$+375$	$a^3b^2eg^4$	$+127500$	$a^3cd^2f^2g^2$	$+750$	$a^2b^3efg^3$
-15	$a^3b^2f^2g^3$	$+30000$	$a^3cde^2fg^2$	-410	$a^2b^3f^3g^2$
$+3000$	a^3bcdg^4	-412500	a^3cdef^3g	-7500	$a^2b^2c^2g^4$
-5550	a^3bcefg^3	$+187500$	a^3cdf^5	$+43500$	$a^2b^2cdfg^3$
$+750$	$a^3bcf^3g^2$	-150000	$a^3ce^4g^2$	$+16875$	$a^2b^2ce^2g^3$
-4800	$a^3bd^2fg^3$	$+412500$	$a^3ce^3f^2g$	-54675	$a^2b^2cef^2g^2$
-27000	$a^3bde^2g^3$	-187500	$a^3ce^2f^4$	$+25500$	$a^2b^2cf^4g$

+ 127500	$a^2b^2d^2eg^3$	+ 1062500	$a^2c^3e^3g^2$	+ 83200	$ab^3d^2f^3g$
− 171300	$a^2b^2d^2f^2g^2$	− 2821875	$a^2c^3e^2f^2g$	+ 37500	$ab^3de^3g^2$
− 346500	$a^2b^2de^2fg^2$	+ 1265625	$a^2c^3ef^4$	+ 511500	$ab^3de^2f^2g$
+ 616500	$a^2b^2def^3g$	− 1350000	$a^2c^2d^3fg^2$	− 288000	ab^3def^4
− 240000	$a^2b^2df^5$	− 3562500	$a^2c^2d^2e^2g^2$	− 202500	ab^3e^4fg
+ 7500	$a^2b^2e^4g^2$	+ 4725000	$a^2c^2d^2ef^2g$	+ 121500	$ab^3e^3f^3$
− 23250	$a^2b^2e^3f^2g$	− 2062500	$a^2c^2d^2f^4$	+ 412500	$ab^2c^3eg^3$
+ 11250	$a^2b^2e^2f^4$	+ 4875000	$a^2c^2de^3fg$	− 23250	$ab^2c^3f^2g^2$
+ 57000	$a^2bc^3fg^3$	− 2625000	$a^2c^2de^2f^3$	+ 675000	$ab^2c^2d^2g^3$
+ 30000	$a^2bc^2deg^3$	− 1875000	$a^2c^2e^5g$	− 3172500	$ab^2c^2defg^2$
− 346500	$a^2bc^2df^2g^2$	+ 1125000	$a^2c^2e^4f^2$	+ 511500	$ab^2c^2df^3g$
− 596250	$a^2bc^2e^2fg^2$	+ 3750000	$a^2cd^4eg^2$	− 2821875	$ab^2c^2e^3g^2$
+ 1222500	$a^2bc^2ef^3g$	− 300000	$a^2cd^4f^2g$	+ 7633125	$ab^2c^2e^2f^2g$
− 506250	$a^2bc^2f^5$	− 9750000	$a^2cd^3e^2fg$	− 3442500	$ab^2c^2ef^4$
− 330000	$a^2bcd^3g^3$	+ 5250000	$a^2cd^3ef^3$	− 2190000	$ab^2cd^3fg^2$
+ 1590000	$a^2bcd^2efg^2$	+ 3750000	$a^2cd^2e^4g$	+ 4725000	$ab^2cd^2e^2g^2$
− 330000	$a^2bcd^2f^3g$	− 2250000	$a^2cd^2e^3f^2$	+ 6030000	$ab^2cd^2ef^2g$
+ 750000	$a^2bcde^3g^2$	− 1000000	$a^2d^6g^2$	− 3360000	$ab^2cd^2f^4$
− 3172500	$a^2bcde^2f^2g$	+ 3000000	a^2d^5efg	− 15337500	ab^2cde^3fg
+ 1537500	a^2bcdef^4	− 1600000	$a^2d^5f^3$	+ 8392500	$ab^2cde^2f^3$
+ 375000	a^2bce^4fg	− 1250000	$a^2d^4e^3g$	+ 5062500	ab^2ce^5g
− 225000	$a^2bce^3f^3$	+ 750000	$a^2d^4e^2f^2$	− 3037500	$ab^2ce^4f^2$
+ 780000	$a^2bd^4fg^2$	+ 9375	ab^4cg^4	− 300000	$ab^2d^4eg^2$
− 1350000	$a^2bd^3e^2g^2$	− 7500	ab^4dfg^3	+ 900000	$ab^2d^3e^2fg$
− 2190000	$a^2bd^3ef^2g$	− 9375	$ab^4e^2g^3$	− 480000	$ab^2d^3ef^3$
+ 1200000	$a^2bd^3f^4$	+ 25500	$ab^4ef^2g^2$	− 375000	$ab^2d^2e^4g$
+ 4650000	$a^2bd^2e^3fg$	− 11520	ab^4f^4g	+ 225000	$ab^2d^2e^3f^2$
− 2550000	$a^2bd^2e^2f^3$	− 97500	$ab^3c^2fg^3$	− 900000	abc^4dg^3
− 1500000	a^2bde^5g	− 412500	ab^3cdeg^3	+ 375000	abc^4efg^2
+ 900000	$a^2bde^4f^2$	+ 616500	$ab^3cdf^2g^2$	− 202500	abc^4f^3g
− 150000	$a^2c^4eg^3$	+ 1222500	$ab^3ce^2fg^2$	+ 4650000	$abc^3d^2fg^2$
+ 7500	$a^2c^4f^2g^2$	− 2197800	ab^3cef^3g	+ 4875000	$abc^3de^2g^2$
+ 250000	$a^2c^3d^2g^3$	+ 864000	ab^3cf^5	− 15337500	abc^3def^2g
+ 750000	$a^2c^3defg^2$	+ 50000	$ab^3d^3g^3$	+ 7087500	abc^3df^4
+ 37500	$a^2c^3df^3g$	− 330000	$ab^3d^2efg^2$	+ 843750	abc^3e^3fg

$$
\begin{array}{lll}
-506250\,abc^3e^2f^3 & +37500\,b^5cfg^3 & -506250\,b^3c^2e^3fg \\
-9750000\,abc^2d^3eg^2 & +187500\,b^5deg^3 & +303750\,b^3c^3e^2f^3 \\
+900000\,abc^2d^3f^2g & -240000\,b^5df^2g^2 & +5250000\,b^3cd^3eg^2 \\
+24750000\,abc^2d^2e^2fg & -506250\,b^5e^2fg^2 & -480000\,b^3cd^3f^2g \\
-13350000\,abc^2d^2ef^3 & +864000\,b^5ef^3g & -13350000\,b^3cd^2e^2fg \\
-9375000\,abc^2de^4g & -331776\,b^5f^5 & +7200000\,b^3cd^2ef^3 \\
+5625000\,abc^2de^3f^2 & -187500\,b^5c^2eg^3 & +5062500\,b^3cde^4g \\
+3000000\,abcd^5g^2 & +11250\,b^4c^2f^2g^3 & -3037500\,b^3cde^3f^2 \\
-9000000\,abcd^4efg & -375000\,b^4cd^2g^3 & -1600000\,b^3d^5g^2 \\
+4800000\,abcd^4f^3 & +1537500\,b^4cdefg^2 & +4800000\,b^3d^4efg \\
+3750000\,abcd^3e^3g & -288000\,b^4cdf^3g & -2560000\,b^3d^4f^3 \\
-2250000\,abcd^3e^2f^2 & +1265625\,b^4ce^3g^2 & -2000000\,b^3d^3e^3g \\
+250000\,ac^6g^3 & -3442500\,b^4ce^2f^2g & +1200000\,b^3d^3e^2f^2 \\
-1500000\,ac^5dfg^2 & +1555200\,b^4cef^4 & -150000\,b^2c^5g^3 \\
-1875000\,ac^5e^2g^2 & +1200000\,b^4d^3fg^2 & +900000\,b^2c^4dfg^2 \\
+5062500\,ac^5ef^2g & -2062500\,b^4d^2e^2g^2 & +1125000\,b^2c^4e^2g^2 \\
-2278125\,ac^5f^4 & -3360000\,b^4d^2ef^2g & -3037500\,b^2c^4ef^2g \\
+3750000\,ac^4d^2eg^2 & +1843200\,b^4d^2f^4 & +1366875\,b^2c^4f^4 \\
-375000\,ac^4d^2f^2g & +7087500\,b^4de^3fg & -2250000\,b^2c^3d^2eg^2 \\
-9375000\,ac^4de^2fg & -3888000\,b^4de^2f^3 & +225000\,b^2c^3d^2f^2g \\
+5062500\,ac^4def^3 & -2278125\,b^4e^5g & +5625000\,b^2c^3de^2fg \\
+3515625\,ac^4e^4g & +1366875\,b^4e^4f^2 & -3037500\,b^2c^3def^3 \\
-2109375\,ac^4e^3f^2 & +500000\,b^3c^3dg^3 & -2109375\,b^2c^3e^4g \\
-1250000\,ac^3d^4g^2 & -225000\,b^3c^3efg^2 & +1265625\,b^2c^3e^3f^2 \\
+3750000\,ac^3d^3efg & +121500\,b^3c^3f^3g & +750000\,b^2c^2d^4g^2 \\
-2000000\,ac^3d^3f^3 & -2550000\,b^3c^2d^2fg^2 & -2250000\,b^2c^2d^3efg \\
-1562500\,ac^3d^2e^3g & -2625000\,b^3c^2de^2g^2 & +1200000\,b^2c^2d^3f^3 \\
+937500\,ac^3d^2e^2f^2 & +8392500\,b^3c^2def^2g & +937500\,b^2c^2d^2e^3g \\
-3125\,b^6g^4 & -3888000\,b^3c^2df^4 & -562500\,b^2c^2d^2e^2f^2
\end{array}
$$

257. If it be required to determine a cubic whose covariant $\overline{12}^3$ with a sextic vanishes identically, it will be found that the problem furnishes linear equations enough to eliminate all coefficients of the cubic, and the condition to be satisfied is, that the determinant B of the sextic vanish. But if the problem were to determine a quartic whose $\overline{12}^3$ with the sextic should

vanish identically; though there are enough linear equations to give a determinant condition among the coefficients of the sextic, this will be found skew symmetrical of the fifth order, and therefore the condition is identically satisfied. Hence, this problem admits of solution generally. In fact, it is easy to verify that if we write the quartic S, which we shall now call

$$i=(ae-4bd+3c^2)x^4+2(af-3be+2cd)x^3y+(ag-9ce+8d^2)x^2y^2+\&c.$$
$$=\iota_0x^4+4\iota_1x^3y+6\iota_2x^2y^2+4\iota_3xy^3+\iota_4y^4,$$

then
$$a\iota_3-3b\iota_2+3c\iota_1-d\iota_0=0, \&c.,$$

thus it satisfies the conditions of the question.*

This quartic covariant furnishes to the system of the sextic besides its invariants its Hessian and its sextic covariant.

The determinant value of B, treated by the rule of Art. 219*b*, at once gives the invariant

$$A^2-36B=12(\iota_0\iota_4-4\iota_1\iota_3+3\iota_2^2)=12I_2,$$

as we shall write for brevity. If the coefficients of the Hessian of i be written out, we find that its source satisfies the relation

$$12(\iota_0\iota_2-\iota_1^2)+A\iota_0=3(al_2-2bl_1+cl_0).$$

Writing out the corresponding equations for the other coefficients, multiplying the first by ι_4 the second by $-4\iota_3$, and so on, and adding, also calling the invariant

$$\iota_0\iota_2\iota_4+2\iota_1\iota_2\iota_3-\iota_0\iota_3^2-\iota_4\iota_1^2-\iota_2^3=I_3,$$

---

* It should have been noticed that if in a skew symmetrical determinant of *odd* order the constituents in any row or column be replaced by arbitrary quantities, the determinant formed is a linear function of the binary determinants formed from the constituents of the replaced and its conjugate column or row. And again, if in a skew symmetrical determinant of *even* order all the constituents in any row or column be replaced by arbitrary quantities, the value of the determinant formed is the product of two linear functions of the constituents of the replaced and its conjugate column or row. Thus the value of the determinant

$$\begin{vmatrix} 0, & a, & -3b, & 3c, & -d, & b_0 \\ -a, & 0, & 6c, & -8d, & 3e, & -4b_1 \\ 3b, & -6c, & 0, & 6e, & -3f, & 6b_2 \\ -3c, & 8d, & -6e, & 0, & g, & -4b_3 \\ d, & -3e, & 3f, & -g, & 0, & b_4 \\ a_0, & -4a_1, & 6a_2, & -4a_3, & a_4, & 0 \end{vmatrix} = 6^2(a_0\iota_4-4a_1\iota_3+6a_2\iota_2-4a_3\iota_1+a_4\iota_0)\cdot(b_0\iota_4-4b_1\iota_3+6b_2\iota_2-4b_3\iota_1+b_4\iota_0),$$

enables us to write down in a determinant of the fifth order the product of the covariant i by any one of its coefficients.

we get $36I_3 + 2AI_2 = 12(l_0l_2 - l_1^2) = 3(4AB + 3C)$, Art. 253,
whence $108AB + 54C - A^3 = 216I_3$.

The relations just employed show that the Hessian of i may be replaced in the system of the sextic by the result of operating with l on the sextic, and the sextic covariant of i by the result of operating with l on the Jacobian of the sextic and i.

The same relations being employed to determine the quadrinvariant of the function $(al_2 - 2bl_1 + cl_0)x^4 + \&c.$, lead to the equation

$$(\iota_0 l_2 - 2\iota_1 l_1 + \iota_2 l_0) l_2 - 2(\iota_1 l_2 - 2\iota_2 l_1 + \iota_3 l_0) l_1 + (\iota_2 l_2 - 2\iota_3 l_1 + \iota_4 l_0) l_0$$
$$= \tfrac{2}{3}(2I_2^2 + 3AI_3) = \tfrac{1}{6}(2A^2B + 3AC + 72B^2).$$

258. It should have been mentioned in dealing with the system of a quartic and a quadric that if we call the quadric covariants at the end of p. 203 respectively

$$\alpha_1 x^2 + 2\beta_1 xy + \gamma_1 y^2, \quad \alpha' x^2 + 2\beta' xy + \gamma' y^2,$$

and operate with the former on the quartic a new quadric covariant $\alpha_2 x^2 + 2\beta_2 xy + \gamma_2 y^2$ is found, if we operate with this on the quartic we get another $\alpha_3 x^2 + 2\beta_3 xy + \gamma_3 y^2$, and so on.

From the system of equations thus derived

$$\alpha_1 = a\gamma - 2b\beta + c\alpha, \quad \alpha_2 = a\gamma_1 - 2b\beta_1 + c\alpha_1, \quad \alpha_3 = a\gamma_2 - 2b\beta_2 + c\alpha_2,$$
$$\beta_1 = b\gamma - 2c\beta + d\alpha, \quad \beta_2 = b\gamma_1 - 2c\beta_1 + d\alpha_1, \quad \beta_3 = b\gamma_2 - 2c\beta_2 + d\alpha_2,$$
$$\gamma_1 = c\gamma - 2d\beta + e\alpha, \quad \gamma_2 = c\gamma_1 - 2d\beta_1 + e\alpha_1, \quad \gamma_3 = c\gamma_2 - 2d\gamma_2 + e\alpha_2,$$

it can easily be seen that each covariant admits of linear expression by two preceding it in the series, the values being $\alpha_3 = S\alpha_1 + 2T\alpha$, $\alpha_4 = S\alpha_2 + 2T\alpha_1$, &c. Moreover, these equations show that $\alpha\gamma_2 - 2\beta\beta_2 + \gamma\alpha_2 = 2(\alpha_1\gamma_1 - \beta_1^2)$, and when we write

$$S\alpha_1 + 2T\alpha = a\gamma_2 - 2b\beta_2 + c\alpha_2, \ \&c.,$$

we see that

$$\alpha_1\gamma_2 - 2\beta_1\beta_2 + \gamma_1\alpha_2 = S(\alpha\gamma_1 - 2\beta\beta_1 + \gamma\alpha_1) + 4T(\alpha\gamma - \beta^2),$$
$$\alpha_2\gamma_2 - \beta_2^2 = S(\alpha_1\gamma_1 - \beta_1^2) + T(\alpha\gamma_1 - 2\beta\beta_1 + \gamma\alpha_1).$$

In the notation of Art. 212, $4(\alpha\gamma - \beta^2) = \Sigma'$, $\alpha\gamma_1 - 2\beta\beta_1 + \gamma\alpha_1 = \phi$, $\alpha_1\gamma_1 - \beta_1^2 = \Sigma + S(\alpha\gamma - \beta^2)$, thus the successive invariants of the quadrics are expressed in terms of these five invariants, and as the skew invariant differs only by a factor from $\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix}$, its square can be similarly expressed.

It may also be noticed as regards the covariant α' that we find $3\alpha_2 = 2S\alpha + \alpha'$; and likewise for the series of derivatives according to the law corresponding to α' from the quartic,

$$\alpha''' = 3S^2\alpha' + 2(54T^2 - S^3)\alpha, \text{ \&c.}$$

Also $$\alpha'' + S\alpha' = 2S^2\alpha + 18T\alpha_1.$$

259. Now when we take as quartic the covariant i of a sextic and take its covariant l as quadric, we derive a new quadric m by operating with l on i, another n by operating with m on i, and by what we have just seen, the further quadric covariants thus derived are reducible. These three thus give rise to a complete quadric system of which we proceed to consider the invariants.

The relations between the coefficients of i and of its Hessian used in Art. 257 give

$$12\{a(\iota_2\iota_4 - \iota_3^2) - 2b(\iota_1\iota_4 - \iota_2\iota_3) + c(\iota_0\iota_4 + 2\iota_1\iota_3 - 3\iota_2^2) - 2d(\iota_0\iota_3 - \iota_1\iota_2) + e(\iota_1\iota_2 - \iota_1^2)\} =$$

$$6(\iota_0 l_2 - 2\iota_1 l_1 + \iota_2 l_0) = m_0,*$$

$$6(\iota_1 l_2 - 2\iota_2 l_1 + \iota_3 l_0) = m_1,$$

$$6(\iota_2 l_2 - 2\iota_3 l_1 + \iota_4 l_0) = m_2,$$

m_1, m_2 having similar expressions to that for m_0.

By the same relations, determining the result of operating with the square of l on the Hessian of i, we find this invariant linearly related to the result of operating on the sextic with the cube of l, $al_2^3 - 6bl_2^2l_1 + \text{\&c.} = D_1$, and to the last invariant written in Art. 257; but it is also obviously so related to the discriminants of l and of m. We find as the result of the comparison what we shall call

$$D' = m_0m_2 - m_1^2 = 9(D_1 - 24BI_3).\dagger$$

---

* In full, the value of $m_0 =$

$$a^2cg^2 - 6a^2dfg + 8a^2e^2g - 3a^2ef^2 - ab^2g^2 + 6abcfg + 27ac^2eg - 34abdeg$$
$$- 4acd^2g + 48abdf^2 - 45ac^2f^2 - 18abe^2f + 78acdef - 48ad^3f - 36ace^3$$
$$+ 28ad^2e^2 - 9b^2ceg + 72b^2d^2g - 96bc^2dg + 36c^4g - 144b^2def + 108bc^2ef$$
$$+ 96bcd^2f - 72c^3df + 81b^2e^3 - 126bcde^2 - 27c^3d^2 + 16bd^3e + 96c^2d^2e - 32cd^4.$$

† It was mentioned in the second edition that instead of the discriminant Δ we may use another invariant of the tenth order D, in which no higher power than the fourth of the extreme coefficients a, g appears, and which does not contain the product a^4g^4. The quantity multiplying a^4 in D is $(eg - f^2)^3$, and the relation

We can now, writing $6(\iota_0 m_2 - 2\iota_1 m_1 + \iota_2 m_0) = n_0$, &c., express the remaining invariants of the quadratics by means of the invariants A, I_2, I_3, D'. In fact, using last article, we have

$$l_0 n_2 - 2l_1 n_1 + l_2 n_0 = 2(m_0 m_2 - m_1^2) = 2D',$$

$$m_0 n_2 - 2m_1 n_1 + m_2 n_0 = 36\{I_2(l_0 m_2 - 2l_1 m_1 + l_2 m_0) + 24I_3(l_0 l_2 - l_1^2)\},$$

$$n_0 n_2 - n_1^2 = 36\{I_2(m_0 m_2 - m_1^2) + 6I_3(l_0 m_2 - 2l_1 m_1 + l_2 m_0)\},$$

and we have already at the end of Article 257,

$$l_0 m_2 - 2l_1 m_1 + l_2 m_0 = 4(2I_2^2 + 3AI_3).$$

We have, of course, also the skew invariant of the system of quadrics which must be to a numerical factor the E of the sextic, and its square will be expressed by the other invariants in the usual manner by putting in their values in the expanded formula for

$$\begin{vmatrix} l_0, & l_1, & l_2 \\ m_0, & m_1, & m_2 \\ n_0, & n_1, & n_2 \end{vmatrix} \begin{vmatrix} l_2, & -2l_1, & l_0 \\ m_2, & -2m_1, & m_0 \\ n_2, & -2n_1, & n_0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{3}(AI_2 + 18I_3), & 4(2I_2^2 + 3AI_3), & 2D' \\ 4(2I_2^2 + 3AI_3), & 2D', & 288(I_2^3 + 2AI_2I_3 + 9I_3^2) \\ 2D', & 288(I_2^3 + 2AI_2I_3 + 9I_3^2), & 72(I_2D' + 48I_2^2I_3 + 72AI_3^2) \end{vmatrix};$$

moreover, the identical relation between the quadrics may be written down by equating this determinant bordered with them to zero.

260. By means of the differential equation I calculated the invariant E. Its value was given at length in the second edition, where it occupied thirteen pages, but I have not thought it worth while to reprint so long a formula. The terms containing the highest power of a are

$$a^6(eg - f^2)^3(g^2 d - 3gfe + 2f^3);$$

whence making all coefficients vanish but a and f the determinant at end of last article is found $= 2(27E)^2$.

---

connecting Δ and D is $\Delta = A^5 - 375A^3B - 625A^2C + 3125D$. The value of D was there given at length, but I have not thought necessary to reprint it. Taking either of its special values in the cases mentioned at the beginning of Art. 256, D_1 is found connected with it by the equation $D_1 = D + 6B(3C + 2AB)$: whence we find

$$9\Delta - 3125D' = 384A^5 - 12000A^2(AI_2 + 5I_3) + 75000I_2(AI_2 + 6I_3).$$

The expression for E in terms of the other invariants may be otherwise found from the following considerations: If in the sextic b, d, f vanish, E necessarily vanishes. For, since the weight of E is forty-five (Art 143), the weight of some one of the constituent coefficients in each term must be expressed by an odd number; and when in the sextic we make all the terms vanish whose weight is odd, E vanishes. $E=0$ is therefore the condition that the roots of the sextic should form a system in involution. If then we make $b, d, f=0$ in A, B, C, Δ and eliminate a, c, e, g from the results, the relation thus obtained between A, B, C, Δ must be satisfied when E vanishes, and must therefore contain it as a factor.

From what has been just remarked it follows that the expression for E in terms of the roots is the product of the fifteen determinants of the form

$$\begin{vmatrix} 1, & \alpha+\beta, & \alpha\beta \\ 1, & \gamma+\delta, & \gamma\delta \\ 1, & \varepsilon+\phi, & \varepsilon\phi \end{vmatrix},$$

or of the fifteen factors (Ex. 7, p. 25)

$$(\alpha-\delta)(\gamma-\phi)(\varepsilon-\beta)+(\beta-\gamma)(\delta-\varepsilon)(\phi-\alpha).$$

If we write $ag=\lambda$, $ce=\mu$, $ae^3+gc^3=\nu$, the values of the invariants got by making $b, d, f=0$, may be written

$$A=\lambda+15\mu, \quad B=\lambda\mu+\mu^2-\nu,$$

$$C=-24\lambda\mu^2-8\mu^3+4(\lambda+3\mu)\nu,$$

$$\Delta=\lambda\{\lambda^2-150\lambda\mu-1875\mu^2+500\nu\}^2.$$

Eliminating ν in the first place, the last two equations become

$$C=4\mu(\lambda-\mu)^2-4(\lambda+3\mu)B, \quad \Delta=\lambda(\lambda^2+350\lambda\mu-1375\mu^2-500B)^2.$$

Then eliminating μ by the help of the first equation, we get

$$1024\lambda^5-1152\lambda^3A+(132A^2-10800B)\lambda+3375C+2700AB-4A^3=0,$$

$$\lambda(256\lambda^2-320A\lambda+55A^2+4500B)^2-\Delta=0.$$

The resultant of these two equations is of the thirtieth degree in the coefficients; and therefore, from what we have seen, can only differ by a constant multiplier from E^2.

261. By Art. 234, when the discriminant is negative the sextic has either six or two real roots; and when it is positive, has either four or none. We can readily anticipate that the discussion of this expression for E is likely to lead to the same results in affording criteria for further distinguishing these cases, as the corresponding discussion of the expression G in the case of the quintic. Analogy also leads us to expect that what will be important to examine will be the result of making $\Delta = 0$ in the expression for E. Now, although the calculation of this general expression for E may be a little laborious, that part of it which is independent of Δ is easily obtained. It will evidently be the product of $3375C + 2700AB - 4A^3$ by the square of the resultant of the cubic and of the quadratic

$$256\lambda^2 - 320A\lambda + 55A^2 + 4500B.$$

And again, analogy leads us to believe that the first of these factors is not important in the question of the criteria for real roots, and that it is the square factor alone which needs to be attended to.

The result I find is that, writing for convenience B' for $100B$, C' for $125C$, the quantity squared differs only by a constant multiplier from

$$4A^6 - 19A^4B' - 49A^2B'^2 - 4A^3C' - 80B'^3 + 52AB'C' - 4C'^2.$$

Analogy then leads me to suppose that the criteria for the number of real roots of a sextic depend on the signs of this quantity, and of $A^2 - 100B$, $A^3 - 125C$,

$$(A^3 - 300AB + 250C)^2 - 5(A^2 - 100B)^3,$$

which, as we saw, vanish when three roots are all equal.

262. If we now resume, from Art. 223, the consideration of the system of two quartics

$$u = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$
$$v = a'x^4 + 4b'x^3y + 6c'x^2y^2 + 4d'xy^3 + e'y^4;$$

and, as in Art. 216, write their Jacobian or functional determinant, which is in full,

$$\begin{aligned}(ab')x^6 + 3(ac')x^5y + 3\{(ad') + 2(bc')\}x^4y^2 + \{(ae') + 8(bd')\}x^3y^3 \\ + 3\{(be') + 2(cd')\}x^2y^4 + 3(ce')xy^5 + (de')y^6 \\ = a_0x^6 + 6a_1x^5y + 15a_2x^4y^2 + 20a_3x^3y^3 + 15a_4x^2y^4 + 6a_5xy^5 + a_6y^6,\end{aligned}$$

and their quadric combinant of the same order in the coefficients

$$\{(ad') - 3(bc')\}x^2 + \{(ae') - 2(bd')\}xy + \{(be') - 3(cd')\}y^2$$
$$= p_0x^2 + 2p_1xy + p_2y^2,$$

we express as follows all the determinants of the second order employed in Arts. 220–1 in calculating the combinant invariants,

$$(ab') = a_0,$$
$$(ac') = 2a_1, \quad (ad') = 3a_2 + \tfrac{2}{5}p_0, \quad (bc') = a_2 - \tfrac{1}{5}p_0,$$
$$(ce') = 2a_5, \quad (ae') = 4a_3 + \tfrac{8}{5}p_1, \quad (bd') = 2a_3 - \tfrac{1}{5}p_1,$$
$$(de') = a_6, \quad (be') = 3a_4 + \tfrac{2}{5}p_2, \quad (cd') = a_4 - \tfrac{1}{5}p_2.$$

Between these determinants we have five identical relations of the form $(ab')(cd') + (ac')(db') + (ad')(bc') = 0$, but of which any two result from the other three, and these introduce the simplest quartic covariant of our sextic Jacobian. In fact, writing as in Art. 257,

$$a_0a_4 - 4a_1a_3 + 3a_2^2 = \iota_0, \quad a_0a_5 - 3a_1a_4 + 2a_2a_3 = 2\iota_1, \text{ &c.},$$

the identities are

$$a_0p_2 - 2a_1p_1 + a_2p_0 = 5\iota_0 - \tfrac{2}{5}p_0^2,$$
$$a_1p_2 - 2a_2p_1 + a_3p_0 = 5\iota_1 - \tfrac{2}{5}p_0p_1,$$
$$a_2p_2 - 2a_3p_1 + a_4p_0 = 5\iota_2 - \tfrac{2}{1\,5}(p_0p_2 + 2p_1^2),$$
$$a_3p_2 - 2a_4p_1 + a_5p_0 = 5\iota_3 - \tfrac{2}{5}p_1p_2,$$
$$a_4p_2 - 2a_5p_1 + a_6p_0 = 5\iota_4 - \tfrac{2}{5}p_2^2.$$

Now if, in order to combine these, we write, as in Art. 252,

$$a_0\iota_4 - 4a_1\iota_3 + 6a_2\iota_2 - 4a_3\iota_1 + a_4\iota_0 = 2l_0,$$
$$a_1\iota_4 - 4a_2\iota_3 + 6a_3\iota_2 - 4a_4\iota_1 + a_5\iota_0 = 2l_1,$$
$$a_2\iota_4 - 4a_3\iota_3 + 6a_4\iota_2 - 4a_5\iota_1 + a_6\iota_0 = 2l_2;$$

also calling

$$a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2 = A_2, \qquad p_0p_2 - p_1^2 = P,$$
$$a_0p_2^2 - 4a_1p_1p_2 + 2a_2(p_0p_2 + 2p_1^2) - 4a_3p_1p_0 + a_4p_0^2 = [a_0p_2^2],$$

we find $2\{\iota_0p_2 - 2\iota_1p_1 + \iota_2p_0 + \tfrac{1}{3}A_2p_0\} = 5.2.l_0 - \tfrac{2}{5}[a_0p_2^2]$,

$$[a_0p_2^2] = 5(\iota_0p_2 - 2\iota_1p_1 + \iota_2p_0) - \tfrac{8}{1\,5}Pp_0,$$

from which eliminating $[a_0 p_2^2]$, also writing down similar relations for the following coefficients and putting for brevity $K = \frac{1}{6} A_2 - \frac{4}{75} P$, we get

$$\iota_0 p_2 - 2\iota_1 p_1 + \iota_2 p_0 + K p_0 = \tfrac{5}{2} l_0,$$
$$\iota_1 p_2 - 2\iota_2 p_1 + \iota_3 p_0 + K p_1 = \tfrac{5}{2} l_1,$$
$$\iota_2 p_2 - 2\iota_3 p_1 + \iota_4 p_0 + K p_2 = \tfrac{5}{2} l_2.$$

It will be observed that the ι and l coefficients depend solely on the coefficients a, hence these equations enable us to determine the p coefficients by means of those of the sextic Jacobian. In fact, they enable us to solve linearly for the p coefficients in terms of these quantities and P, and when this is done to form from them the value of P.

Thus the notation of Arts. 256, 258 gives us the determinant

$$\begin{vmatrix} \iota_0 , & 2\iota_1 , & \iota_2 + K \\ \iota_1 , & 2\iota_2 - K, & \iota_3 \\ \iota_2 + K, & 2\iota_3 , & \iota_4 \end{vmatrix} = 2I_3 - I_2 K + K^3,$$

we find incidentally $50 I_2 + 4KP = 15\,(l_0 p_2 - 2 l_1 p_1 + l_2 p_0)$, and finally to determine $P = \frac{25}{8}\,(A_2 - 6K)$ arrive at the quintic*

$$6K^5 - A_2 K^4 - 10 I_2 K^3 + 2\,(A_2 I_2 + 15 I_3)\,K^2 - 8 A_2 I_3 K$$
$$- 26 I_2 I_3 - A_2 I_2^2 + \tfrac{1}{6} D' = 0.$$

Hence we see that as the Jacobian is a sextic of full generality, it is possible to express any given sextic as the functional determinant of two quartics in five ways; when any root of this quintic is employed, a quadric p is linearly found by means of the given sextic by the above equations, or we have definitely in terms of the covariants l, m, n of the sextic:

$$\tfrac{2}{5}\,(2I_3 - I_2 K + K^3)\,p = (K^2 - I_2)\,l - \tfrac{1}{6} K m + \tfrac{1}{36} n,$$

and hence the values of the determinants (ad'), &c., are found.

* In preparing the following sketch of Dr. Brill's paper, *Math. Annalen* XX., p. 330, I had printed thus far without seeing the "Mémoire sur les faisceaux de formes binaires ayant une même Jacobienne," by M. Cyparissos Stephanos, in Tome XXVII. of the *Savants Etrangers* of the Académie des Sciences, 1883. M. Stephanos obtains this quintic, but with a slightly different notation, p. 78. In the preliminary notice in the *Comptes Rendus*, 12 Dec., 1881, it is given incorrectly.

263. From the system of three quartics

$U=\alpha x^4+4\beta x^3y+6\gamma x^2y^2+4\delta xy^3+\varepsilon y^4$, $V=\alpha_{,}x^4+\ldots$, $W=\alpha' x^4+\ldots$,

we can obtain a similar theory. In fact, they afford the sextic covariant

$$\begin{vmatrix} \alpha x^2+2\beta xy+\gamma y^2, & \beta x^2+2\gamma xy+\delta y^2, & \gamma x^2+2\delta xy+\varepsilon y^2 \\ \alpha_{,}x^2+2\beta_{,}xy+\gamma_{,}y^2, & \beta_{,}x^2+2\gamma_{,}xy+\delta_{,}y^2, & \gamma_{,}x^2+2\delta_{,}xy+\varepsilon_{,}y^2 \\ \alpha' x^2+2\beta' xy+\gamma' y^2, & \beta' x^2+2\gamma' xy+\delta' y^2, & \gamma' x^2+2\delta' xy+\varepsilon' y^2 \end{vmatrix},$$

which, using $(\alpha\beta\gamma)$ to denote the determinant $(\alpha\beta_{,}\gamma')$, &c., we may write

$$=(\alpha\beta\gamma)x^6+2(\alpha\beta\delta)x^5y+[(\alpha\beta\varepsilon)+3(\alpha\gamma\delta)]x^4y^2+2[(\alpha\gamma\varepsilon)+2(\beta\gamma\delta)]x^3y^3$$
$$+[(\alpha\delta\varepsilon)+3(\beta\gamma\varepsilon)]x^2y^4+2(\beta\delta\varepsilon)xy^5+(\gamma\delta\varepsilon)y^6$$
$$=b_0x^6+6b_1x^5y+15b_2x^4y^2+20b_3x^3y^3+15b_4x^2y^4+6b_5xy^5+b_6y^6.$$

Introducing the present notation, we find the expansion (p. 31) of the determinant

$$\begin{vmatrix} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ \alpha, & \beta, & \gamma, & \delta, & \varepsilon, \\ \alpha_{,}, & \beta_{,}, & \gamma_{,}, & \delta_{,}, & \varepsilon_{,} \\ \alpha', & \beta', & \gamma', & \delta', & \varepsilon' \end{vmatrix}$$

$$=a_0b_6-6a_1b_5+15a_2b_4-20a_3b_3+15a_4b_2-6a_5b_1+a_6b_0$$
$$-\tfrac{1}{6}p_0\{(\alpha\delta\varepsilon)-2(\beta\gamma\varepsilon)\}+\tfrac{1}{6}p_1\{(\alpha\gamma\varepsilon)-8(\beta\gamma\delta)\}-\tfrac{1}{6}p_2\{(\alpha\beta\varepsilon)-2(\alpha\gamma\delta)\}.$$

This introduces a new combinant of the three quartics, which we shall write $q_0x^2+2q_1xy+q_2y^2=$

$$[(\alpha\beta\varepsilon)-2(\alpha\gamma\delta)]x^2+[(\alpha\gamma\varepsilon)-8(\beta\gamma\delta)]xy+[(\alpha\delta\varepsilon)-2(\beta\gamma\varepsilon)]y^2;$$

by whose coefficients along with those of b we can express all the determinants*

$$\alpha_6=(\alpha\beta\gamma)=b_0,\quad \alpha_5=(\alpha\beta\delta)=3b_1,\quad \alpha_1=(\beta\delta\varepsilon)=3b_5,\quad \alpha_0=(\gamma\delta\varepsilon)=b_6,$$
$$\alpha_4=(\alpha\gamma\delta)=3b_2-\tfrac{1}{6}q_0,\quad \alpha_3=(\beta\gamma\delta)=b_3-\tfrac{1}{6}q_1,\quad \alpha_2=(\beta\gamma\varepsilon)=3b_4-\tfrac{1}{6}q_2,$$
$$\beta_4=(\alpha\beta\varepsilon)=6b_2+\tfrac{3}{6}q_0,\quad \beta_3=(\alpha\gamma\varepsilon)=8b_3+\tfrac{2}{6}q_1,\quad \beta_2=(\alpha\delta\varepsilon)=6b_4+\tfrac{3}{6}q_2.$$

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* It is useful to have both notations: the identities following may be written by letting x and y go through all pairs of values of $\alpha, \beta, \gamma, \delta, \varepsilon$ in the matrix

$$\begin{Vmatrix} \alpha, & \alpha_{,}, & \alpha', & (xy\alpha) \\ \beta, & \beta_{,}, & \beta', & (xy\beta) \\ \gamma, & \gamma_{,}, & \gamma', & (xy\gamma) \\ \delta, & \delta_{,}, & \delta', & (xy\delta) \\ \varepsilon, & \varepsilon_{,}, & \varepsilon', & (xy\varepsilon) \end{Vmatrix}$$

and are $\beta_2\alpha_2-\beta_3\alpha_1+\beta_4\alpha_0=0$,
$\alpha_1\alpha_4-\alpha_6\alpha_5-\beta_2\alpha_3=0$,
$\alpha_2\alpha_4-\alpha_6\alpha_6-\beta_3\alpha_3=0$,
$\alpha_2\alpha_5-\alpha_1\alpha_6-\beta_4\alpha_3=0$,
$\beta_2\alpha_6-\beta_3\alpha_5+\beta_4\alpha_4=0$.

Among these there are five identities, as in Art. 28, three of which only are independent, and which lead to the system of equations of the type

$$b_0 q_2 - 2b_1 q_1 + b_2 q_0 = \tfrac{1}{5} q_0^2 - 10\,(b_0 b_4 - 4b_1 b_3 + 3b_2^2),$$

whence the determination of the covariant q from b, by aid of which any given sextic may be identified with the functional determinant of three quartics, follows immediately as in last article. It is to be observed that the duality, of which these two articles furnish an example, is general; that if we have a system of p independent quantics of the n^{th} degree, n not less than p, they have a functional determinant of the degree $(n-p+1)\,p$, which is a quantic of full generality of that degree, and that any quantic of the degree $(n-p+1)\,p$ may be identified, by adjoining to it a suitable irrationality, either with the functional determinant of p quantics of the n^{th} degree or with the functional determinant of $n-p+1$ quantics of the n^{th} degree. The irrationality in the present case for either form is the root of a quintic, by means of which the quadratic covariant is linearly determined. Moreover, the system of combinants of the p quantics of the n^{th} order, for n not less than p, is in number and form identical with that of $n-p+1$ quantics of the same order.

264. If the quartic $\lambda U + \mu V + \nu W$ break up into the quadratic factors

$$(r_0 x^2 + 2r_1 xy + r_2 y^2)\,(r_0' x^2 + 2r_1' xy + r_2' y^2),$$

on comparing coefficients, we can find easily that the second factor may be determined, in general, by the first, as

$$\begin{vmatrix} \alpha, & \alpha_1, & \alpha', & r_0, & 0, & 0 \\ 4\beta, & 4\beta_1, & 4\beta', & 2r_1, & r_0, & 0 \\ 6\gamma, & 6\gamma_1, & 6\gamma', & r_2, & 2r_1, & r_0 \\ 4\delta, & 4\delta_1, & 4\delta', & 0, & r_2, & 2r_1 \\ \varepsilon, & \varepsilon_1, & \varepsilon', & 0, & 0, & r_2 \\ 0, & 0, & 0, & x^2, & xy, & y^2 \end{vmatrix},$$

whence this particular value of $\lambda U + \mu V + \nu W$ is found by replacing the bottom row by U, V, W, 0, 0, 0, and accordingly $\lambda : \mu : \nu$ are determined by $r_0 : r_1 : r_2$.

The second quadratic factor becomes indeterminate if the former make all determinants vanish in the matrix

$$\left\| \begin{matrix} \alpha, & \alpha_{,}, & \alpha', & r_0, & 0, & 0 \\ 4\beta, & 4\beta_{,}, & 4\beta', & 2r_1, & r_0, & 0 \\ 6\gamma, & 6\gamma_{,}, & 6\gamma', & r_2, & 2r_1, & r_0 \\ 4\delta, & 4\delta_{,}, & 4\delta', & 0, & r_2, & 2r_1 \\ \varepsilon, & \varepsilon_{,}, & \varepsilon', & 0, & 0, & r_2 \end{matrix} \right\|,$$

the former factor accordingly fails to determine the ratios $\lambda : \mu : \nu$.

The equations which make this matrix vanish may be written

$$b_0 r_2^2 - 4b_1 r_2 r_1 + 2b_2 (r_0 r_2 + 2r_1^2) - 4b_3 r_1 r_0 + b_4 r_0^2 = \tfrac{2}{5} (r_0 r_2 - r_1^2) q_0 - \tfrac{1}{10} (r_0 q_2 - 2r_1 q_1 + r_2 q_0) r_0,$$

$$b_1 r_2^2 - 4b_2 r_2 r_1 + 2b_3 (r_0 r_2 + 2r_1^2) - 4b_4 r_1 r_0 + b_5 r_0^2 = \tfrac{2}{5} (r_0 r_2 - r_1^2) q_1 - \tfrac{1}{10} (r_0 q_2 - 2r_1 q_1 + r_2 q_0) r_1,$$

$$b_2 r_2^2 - 4b_3 r_2 r_1 + 2b_4 (r_0 r_2 + 2r_1^2) - 4b_5 r_1 r_0 + b_6 r_0^2 = \tfrac{2}{5} (r_0 r_2 - r_1^2) q_2 - \tfrac{1}{10} (r_0 q_2 - 2r_1 q_1 + r_2 q_0) r_2,$$

or in the form

$$6r_0^2 \alpha_0 - 8r_0 r_1 \alpha_1 + r_0 r_2 (6\alpha_2 - \beta_2) + 4r_1^2 \beta_2 - \quad 3r_1 r_2 \beta_3 \quad + r_2^2 \beta_4 = 0,$$

$$2r_0^2 \alpha_1 - r_0 r_1 (6\alpha_2 + \beta_2) + 12 r_0 r_2 \alpha_3 + 3r_1^2 \beta_3 - r_1 r_2 (6\alpha_4 + \beta_4) + 2r_2^2 \alpha_5 = 0,$$

$$r_0^2 \beta_2 - 3r_0 r_1 \beta_3 + r_0 r_2 (6\alpha_4 - \beta_4) + 4r_1^2 \beta_4 - \quad 8r_1 r_2 \alpha_5 \quad + 6r_2^2 \alpha_6 = 0;$$

and from these either of the ratios $r_0 : r_1 : r_2$, may be eliminated dialytically and the other will be found by a cubic. A root of this cubic being employed, the former ratio will be found linearly. Thus three quadratics u_1, u_2, u_3 can be found, and we may assume, as the basis* of the system of quartics in all matters relating to combinants, the products $u_2 u_3$, $u_3 u_1$, $u_1 u_2$. If, further, we suppose a linear transformation effected, making the independent variables the factors of u_1, so that $u_1 = xy$ and $r_0 = 0$, $r_2 = 0$, we must have in the equations determining these coefficients $\beta_2 = 0$, $\beta_3 = 0$, $\beta_4 = 0$, whence follow, by the identities of Art. 263 (note) that $\frac{\alpha_1}{\alpha_4} = \frac{\alpha_2}{\alpha_5} = \frac{\alpha_3}{\alpha_6}$. Thus, if we make by suitable determination unity the common value of these quotients, we have by a linear transformation, depending on the solution of the quintic and on that of the cubic, arrived at the reduction of the general sextic to Dr. Brill's canonical form

$$x^6 + 2px^5 + 3qx^4 + 4rx^3 + 3x^2 + 2px + q = 0.$$

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* Compare the Notice of Dr. Brill's paper in the *Fortschritte der Mathematik*, 1882, by Dr. W. F. Meyer; also his *Apolarität und rationale Curven*, p. 305, &c.

265. When the quartic $\lambda U+\mu V+\nu W$ is the square of $s_0x^2+2s_1xy+s_2y^2$, we must have the matrix

$$\left\|\begin{array}{cccc} \alpha, & \alpha_{,}, & \alpha', & s_0^2 \\ \beta, & \beta_{,}, & \beta', & s_0s_1 \\ 3\gamma, & 3\gamma_{,}, & 3\gamma', & s_0s_2+2s_1^2 \\ \delta, & \delta_{,}, & \delta', & s_1s_2 \\ \varepsilon, & \varepsilon_{,}, & \varepsilon', & s_2^2 \end{array}\right\| = 0.$$

But this is equivalent, by the identities of note Art. 263, to only two equations; for example, we may write the two

$$\alpha_3s_0^2-\alpha_4s_0s_1+\tfrac{1}{3}\alpha_5(s_0s_2+2s_1^2)-\alpha_6s_1s_2 \qquad =0,$$
$$\alpha_0s_0s_1-\tfrac{1}{3}\alpha_1(s_0s_2+2s_1^2)+\alpha_2s_1s_2-\alpha_3s_2^2=0,$$

Eliminating $s_0 : s_2$ between these, we get a biquadratic in $s_1 : s_2$, and the second equation connects a single value of $s_0 : s_2$ with each of its roots. Thus four quadratics are found to solve the question: calling them ϕ_1, ϕ_2, ϕ_3, ϕ_4, we may take the squares of three of them as basis of the system as regards combinant properties. It is obvious that, besides the linear relation which holds between any four quadratic functions, there must exist also a linear relation between the squares of ϕ_1, ϕ_2, ϕ_3, ϕ_4.

If $\phi_1=xy$, so that $s_0=0$, $s_2=0$; the vanishing of the above matrix independently of s_1 requires that

$$\alpha_1=0, \quad \alpha_5=0, \quad \beta_2=0, \quad \beta_4=0.$$

In this way we are led, by employing a root of the quintic and a root of the biquadratic, to reduce by linear transformation the general sextic to the following canonical form, given both by M. Stephanos and Dr. Brill,

$$x^6+ax^4+bx^3+cx^2+1=0.$$

266. When, by the methods just explained, the sextic is identified with the Jacobian of a system of quartics, its discriminant breaks up into factors. For the case of two quartics these are (Art. 180) their resultant R and the invariant called D (Art. 221), which vanishes if $u+\lambda v$ admit of a cubic factor. The resultant was already given (p. 220), but we may write it in the notation of Art. 261 as

$$R=12^4\left|\begin{array}{cccc} a_0, & a_1, & a_2+\frac{2}{15}p_0, & a_3+\frac{2}{5}p_1 \\ a_1, & a_2-\frac{4}{45}p_0, & a_3-\frac{2}{45}p_1, & a_4+\frac{2}{15}p_2 \\ a_2+\frac{2}{15}p_0, & a_3-\frac{2}{45}p_1, & a_4-\frac{4}{45}p_2, & a_5 \\ a_3+\frac{2}{5}p_1, & a_4+\frac{2}{15}p_2, & a_5, & a_6 \end{array}\right|.$$

When we reduce this determinant, as can be done very simply by the rule of Art. 219*b*, we find

$$3R = 4^4 \{75K^2 - 20A_2K + 8A_2^2 - 125I_2\}.$$

The invariant D will be found in the same notation by expressing that we have simultaneously

$$a_0x^4 + 4a_1x^3y + (6a_2 - \tfrac{1}{5}p_0)\,x^2y^2 + (4a_3 - \tfrac{2}{5}p_1)\,xy^3 + (a_4 - p_2)\,y^4 = 0,$$
$$a_1x^4 + (4a_2 + \tfrac{1}{5}p_0)\,x^3y + (6a_3 + \tfrac{2}{5}p_1)\,x^2y^2 + (4a_4 + \tfrac{1}{5}p_2)\,xy^3 + a_5y^4 = 0,$$
$$(a_2 - \tfrac{1}{5}p_0)\,x^4 + (4a_3 - \tfrac{2}{5}p_1)\,x^3y + (6a_4 - \tfrac{1}{5}p_2)\,x^2y^2 + 4a_5xy^3 + a_6y^4 = 0,$$

and at once this is written down as a determinant of the sixth order. Dr. Brill shows that the discriminant breaks up into two corresponding factors for the functional determinant of p quantics, and in particular determines their values for three quartics. For his canonical form (Art. 264) it breaks up into a factor of the sixth degree,

$$r^2q^2 + rp(\tfrac{8}{27}p^2 - q)(q^3+1) + (\tfrac{8}{27}p^2 - q)^2p^2 - \tfrac{1}{12}(q^3-1)^2p^2 + \tfrac{1}{4}q(q^3-1)^2 = \tfrac{1}{16}C,$$

and its other factor is the product of four linear factors

$$D_0 = (p-r)^2 + (q-1)^2, \quad D_1 = p + r + q + 1, \quad D_2 = -p - r + q + 1.$$

From these, taking p, q, r as rectangular coordinates, after the manner of Sylvester (p. 242), and considering how space is parted by the surfaces $C=0$, $D_1=0$, $D_2=0$, he investigates the number of real and imaginary roots for real values of p, q, r.

We omit the discussion, however, as well as the geometrical developments given in the Mémoire of M. Stephanos, and conclude with a few miscellaneous examples on the subjects of this Lesson.

Ex. 1. If the three quartics in Art. 263 have a common factor, we may equate them to zero, and, multiplying them first by x and then by y, eliminate x^5, x^4y, &c. Hence we get a determinant

$$\Delta = \begin{vmatrix} \alpha, & 4\beta, & 6\gamma, & 4\delta, & \varepsilon, & 0 \\ \alpha_{,}, & 4\beta_{,}, & 6\gamma_{,}, & 4\delta_{,}, & \varepsilon_{,}, & 0 \\ \alpha', & 4\beta', & 6\gamma', & 4\delta', & \varepsilon', & 0 \\ 0, & \alpha, & 4\beta, & 6\gamma, & 4\delta, & \varepsilon \\ 0, & \alpha_{,}, & 4\beta_{,}, & 6\gamma_{,}, & 4\delta_{,}, & \varepsilon_{,} \\ 0, & \alpha', & 4\beta', & 6\gamma', & 4\delta', & \varepsilon' \end{vmatrix},$$

which we can write in either notation as follows

$$\Delta = 4\{144a_0a_6 - 64a_1a_5 + 24(a_2\beta_4 + a_4\beta_2) + 4\beta_2\beta_4 - 9\beta_3^2\}$$
$$= 48\{8(b_0b_6 - 6b_1b_5 + 15b_2b_4 - 10b_3^2) - \tfrac{1}{3}(q_0q_2 - q_1^2)\}.$$

Ex. 2. If we take as the three quartics in Art. 263, the simplest quartic covariants of the quartics in Art. 262, viz.

$$H = ax^4 + \ldots = (ac - b^2)\, x^4 + \ldots, \quad H_{,} = a_{,}x^4 + \ldots = (ac' + ca' - 2bb')\, x^4 + \ldots,$$
$$H' = a'x^4 + \ldots = (a'c' - b'^2)\, x^4 + \ldots;$$

we can write the invariant C of p. 220 in the form $\begin{vmatrix} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ \alpha, & \beta, & \gamma, & \delta, & \varepsilon \\ \alpha_{,}, & \beta_{,}, & \gamma_{,}, & \delta_{,}, & \varepsilon_{,} \\ \alpha', & \beta', & \gamma', & \delta', & \varepsilon' \end{vmatrix}$.

In fact, when a member of the system $\lambda u + \mu v$ admits of being a perfect square, we can identify $\lambda u + \mu v$ (Art. 207) with $\lambda^2 H + \lambda\mu H_{,} + \mu^2 H'$, and when we do so and eliminate dialytically, we get this determinant.

Ex. 3. In the case of Ex. 2, if $\lambda u + \mu v$ admit of a cube factor, $\lambda^2 H + \lambda\mu H_{,} + \mu^2 H'$ is the fourth power of that factor, whence by the matrix

$$\begin{Vmatrix} \alpha, & \alpha_{,}, & \alpha', & y^4 \\ \beta, & \beta_{,}, & \beta', & -xy^3 \\ \gamma, & \gamma_{,}, & \gamma', & x^2y^2 \\ \delta, & \delta_{,}, & \delta', & -x^3y \\ \varepsilon, & \varepsilon_{,}, & \varepsilon', & x^4 \end{Vmatrix} = 0,$$

we can both determine the factor and the condition $D = 0$.

Ex. 4. The quadric covariant l comes naturally in relation with the sextic covariant T, which may be written in any of the following forms, and which we shall now call $j = j_0x^6 + 6y_1x^5y + \&c.$

$$= \begin{vmatrix} ax^2 + 2bxy + cy^2, & bx^2 + 2cxy + dy^2, & cx^2 + 2dxy + ey^2 \\ bx^2 + 2cxy + dy^2, & cx^2 + 2dxy + ey^2, & dx^2 + 2exy + fy^2 \\ cx^2 + 2dxy + ey^2, & dx^2 + 2exy + fy^2, & ex^2 + 2fxy + gy^2 \end{vmatrix}$$

$$= \begin{vmatrix} ax + by, & bx + cy, & cx + dy, & -y^3 \\ bx + cy, & cx + dy, & dx + ey, & xy^2 \\ cx + dy, & dx + ey, & ex + fy, & -x^2y \\ dx + ey, & ex + fy, & fx + gy, & x^3 \end{vmatrix}$$

$$= \begin{vmatrix} a, & b, & c, & d, & -y^3 \\ b, & c, & d, & e, & xy^2 \\ c, & d, & e, & f, & -x^2y \\ d, & e, & f, & g, & x^3 \\ -y^3, & xy^2, & -x^2y, & x^3, & 0 \end{vmatrix} = \tfrac{1}{4} \begin{vmatrix} a, & b, & c, & -4y^3, & 0 \\ b, & c, & d, & 3y^2x, & -y^3 \\ c, & d, & e, & -2yx^2, & 2y^2x \\ d, & e, & f, & x^3, & -3yx^2 \\ e, & f, & g, & 0, & 4x^3 \end{vmatrix}$$

$$= (abc)\, x^6 + 2\,(abd)\, x^5y + [(abe) + 3\,(acd)]\, x^4y^2 + 2\,\{(ace) + 2\,(bcd)\}\, x^3y^3 + (cde)\, y^6 + 2\,(bde)\, xy^5 + [(ade) + 3\,(bce)]\, x^2y^4,$$

denoting by (abc) &c. the determinants $\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}$ &c. of the matrix $\begin{vmatrix} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \end{vmatrix}$ as above.

In this notation we find

$$l = \{(abe) - 2\,(acd)\}\, x^2 + \{(ace) - 8\,(bcd)\}\, xy + \{(ade) - 2\,(bce)\}\, y^2.$$

But again, let the reciprocal constituents to

$$B = \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$ be written $$\begin{vmatrix} a_{,}, & b_{,}, & c_{,}, & d_{,} \\ b_{,}, & c_{,}', & d_{,}', & e_{,} \\ c_{,}, & d_{,}', & e_{,}', & f_{,} \\ d_{,}, & e_{,}, & f_{,}, & g_{,} \end{vmatrix}$$

and we have

$$\begin{array}{lll} (abc) = j_0 = g_{,} & (abe) = 6j_2 + \tfrac{3}{5}l_0 = e_{,}' - e_{,} & (acd) = 3j_2 - \tfrac{1}{5}l_0 = e_{,} \\ (abd) = 3j_1 = -f_{,} & (ace) = 8j_3 + \tfrac{2}{5}l_1 = -(d_{,}' - d_{,}), & (bcd) = j_3 - \tfrac{1}{5}l_1 = -d_{,} \\ (bde) = 3j_5 = -b_{,} & (ade) = 6j_4 + \tfrac{3}{5}l_2 = c_{,}' - c_{,} & (bce) = 3j_4 - \tfrac{1}{5}l_2 = c_{,}; \\ (cde) = j_6 = a_{,,} & & \end{array}$$

also $e_{,}' = 9j_2 + \tfrac{2}{5}l_0, \quad -d_{,}' = 9j_3 + \tfrac{1}{5}l_1, \quad c_{,}' = 9j_4 + \tfrac{2}{5}l_2.$

Now from the identities between the constituents of B and their reciprocals we can derive relations among the coefficients of covariants, for instance,

$$(aj')_3 = aj_3 - 3bj_2 + 3cj_1 - dj_0 = \tfrac{1}{5}(al_1 - bl_0) = \tfrac{1}{5}(al)_1,$$
$$(aj')_4 = aj_4 - 4bj_3 + 6cj_2 - 4dj_1 + ej_0 = \tfrac{1}{15}(al_2 - 2bl_1 + cl_0) = \tfrac{1}{15}(al)_2,$$
$$(aj')_5 = aj_5 - 5bj_4 + 10cj_3 - 10dj_2 + 5ej_1 - fj_0 = 0,$$
$$(aj')_6 = aj_6 - 6bj_5 + 15cj_4 - 20dj_3 + 15ef_2 - 6fj_1 + gj_0 = 4B.$$

The identities got by writing the above matrix with two additional rows of its own lead to $(Hj)_6 - \tfrac{3}{14}(ij)_4 - \tfrac{1}{5}(il)_2 + \tfrac{1}{30}Al = 0.$

The identities of note Art. 263 applied to the matrix in this Example are

$$\frac{d_{,}(e_{,}' - c_{,})}{e_{,}b_{,} - f_{,}a_{,}} = \frac{d_{,}(d_{,}' - d_{,})}{e_{,}c_{,} - g_{,}a_{,}} = \frac{d_{,}(e_{,}' - e_{,})}{f_{,}c_{,} - g_{,}b_{,}} = 1,$$

and lead to the set of relations of the type

$$j_0j_4 - 4j_1j_3 + 3j_2^2 = \tfrac{1}{30}l_0^2 - \tfrac{1}{10}(l_2j_0 - 2l_1j_1 + l_2j_0).$$

Ex. 5. The definition of l, $ai_4 - 4bi_3 + 6ci_2 - 4di_1 + ei_0 = 2l_0$ &c., combined with the relations $ai_3 - 3bi_2 + 3ci_1 - di_0 = 0$ &c. (Art. 258), give the following identities:

$$\begin{array}{ll} ai_3 - 3bi_2 + 3ci_1 - di_0 = 0, & ai_4 - 3bi_3 + 3ci_2 - di_1 = \tfrac{3}{2}l_0, \\ bi_3 - 3ci_2 + 3di_1 - ei_0 = -\tfrac{1}{2}l_0, & bi_4 - 3ci_3 + 3di_2 - ei_4 = \tfrac{3}{2}l_1, \\ ci_3 - 3di_2 + 3ei_1 - fi_0 = -\tfrac{3}{2}l_1, & ci_4 - 3di_3 + 3ei_2 - fi_1 = \tfrac{1}{2}l_2, \\ di_3 - 3ei_2 + 3fi_1 - gi_0 = -\tfrac{3}{2}l_2, & di_4 - 3ei_3 + 3fi_2 - gi_1 = 0, \end{array}$$

which, being solved for the coefficients of i, lead to the relations

$$Bi_0 = \tfrac{3}{2}(l_0j_2 - 2l_1j_1 + l_2j_0) - \tfrac{1}{10}l_0^2 \text{ &c.},$$

whence, by the last formula of Ex. 4,

$$75(j_0j_4 - 4j_1j_3 + 3j_2^2) + 5Bi_0 = l_0^2 \text{ &c.}$$

Ex. 6. If we compare the values of $\left\| \begin{matrix} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \end{matrix} \right\|$ $\left\| \begin{matrix} e, & -4d, & 6c, & -4b, & a \\ f, & -4e, & 6d, & -4c, & b \\ g, & -4f, & 6e, & -4d, & c \end{matrix} \right\|$,

we find $6(j_0j_6 - 6j_1j_5 + 15j_2j_4 - 10j_3^2) + \tfrac{3}{5}(l_0l_2 - l_1^2) = I_3 - \tfrac{1}{6}AI_2 + \tfrac{1}{24}A^3$ (Art. 257),

whence $30(j_0j_6 - 6j_1j_5 + 15j_2j_4 - 10j_3^2) = 2AB - C.$

Ex. 7. If the three quartics in Art. 263 be the second derived functions of the sextic in Art. 262, the sextic covariant is the j of the sextic $a_0x^6 + \ldots$, and the quadric covariant q is its l. Using the letters j and l in this sense, we have

$$[a_0j_6] = a_0j_6 - 6a_1j_5 + 15a_2j_4 - 20a_3j_3 + 15a_4j_2 - 6a_5j_1 + a_6j_0.$$

$$= 4\begin{vmatrix} a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \\ a_2, & a_3, & a_4, & a_5 \\ a_3, & a_4, & a_5, & a_6 \end{vmatrix},$$

and

$$\begin{vmatrix} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a_0, & a_1, & a_2, & a_3, & a_4 \\ a_1, & a_2, & a_3, & a_4, & a_5 \\ a_2, & a_3, & a_4, & a_5, & a_6 \end{vmatrix} = [a_0j_6] + \tfrac{1}{5}(p_0l_2 - 2p_1l_1 + p_2l_0).$$

Ex. 8. If u, v be any two binary cubics $ax^3+\ldots$, $a'x^3+\ldots$, and if operating with $a\dfrac{d^3}{dy^3}-3b\dfrac{d^3}{dx\,dy^2}+\ldots$ on the product uv, we determine the result to be proportional to u; we find that v must be the evectant of the discriminant of u. This fact has already been employed to establish a canonical form for the sextic (Art. 174).

The determinant of that article may be found, as in Ex. 1, Art. 212, by identifying the cubicovariant $\overline{12}^3$ of the sextic and an arbitrary cubic $ax^3+3\beta x^2y+\ldots$ with this cubic and eliminating its coefficients. Thus

$$\begin{aligned}
a\delta-3b\gamma+3c\beta-d\alpha&=3\rho a,\\
b\delta-3c\gamma+3d\beta-e\alpha&=3\rho\beta,\\
c\delta-3d\gamma+3e\beta-f\alpha&=3\rho\gamma,\\
d\delta-3e\gamma+3f\beta-g\alpha&=3\rho\delta.
\end{aligned}$$

Now the determinant which results,

$$\begin{vmatrix}
a, & b, & c, & d+3\rho\\
b, & c, & d-\rho, & e\\
c, & d+\rho, & e, & f\\
d-3\rho, & e, & f, & g
\end{vmatrix},$$

is unaltered by changing the sign of ρ, whence it is a function of ρ^2 only, and its value is easily found $=B+A\rho^2+9\rho^4$.

The canonical form thus arrived at may be written, with the relation $u+v+w=0$, as

$$au^6+bv^6+cw^6+3duvw\,(v-w)\,(w-u)(u-v)=0.$$

By converting this into a binary system, we find

$$A=bc+ca+ab-9d^2,\quad B=-(bc+ca+ab)\,d^2,$$

whence the determinant just written breaks up into the factors

$$(9\rho^2+bc+ca+ab)\,(\rho^2-d^2).$$

From the value of the covariant

$$i=[bc+(b-c)\,d+3d^2]\,v^2w^2+[ca+(c-a)\,d+3d^2]\,w^2u^2+[ab+(a-b)\,d+3d^2]\,u^2v^2$$

it can easily be found that

$$\begin{aligned}
l=[abc+2da\,(b-c)+d^2\,(5a-b-c)]\,u^2+[abc+2db\,(c-a)+d^2\,(5b-c-a)]\,v^2\\
+[abc+2dc\,(a-b)+d^2\,(5c-a-b)]\,w^2,
\end{aligned}$$

whence by $3C+4AB=4\,(l_0l_2-l_1^2)=3a^2b^2c^2+2d^2\{5abc\,(a+b+c)-2\,(b^2c^2+c^2a^2+a^2b^2)\}$

$$+12d^3\,(b-c)\,(c-a)\,(a-b)-9d^4\{a^2+b^2+c^2-2\,(bc+ca+ab)\},$$

we get $C=a^2b^2c^2+6abc\,(a+b+c)\,d^2+4\,(b-c)\,(c-a)\,(a-b)\,d^3-3\,(a+b+c)^2\,d^4$.

The binary sextic has just been expressed by the intersections of a line with a ternary sextic. The change to expressing it by the intersections of a ternary quadric and cubic may be made thus. Let $u^2=x$, $v^2=y$, $w^2=z$; then, if $2vw=x-y-z$, $2wu=y-z-x$, $2uv=z-x-y$, the relation $u+v+w=0$ becomes the single relation $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$, and the sextic becomes

$$ax^3+by^3+cz^3-3d\,(y-z)\,(z-x)\,(x-y)=0.$$

Comparing with *Curves*, Arts. 220–1, we have the values for this of the ternary invariants, $S=-B$ and $T=C$, whence the discriminant of the ternary cubic is C^2-64B^3.

Ex. 9. The invariants of the system of quadrics p, l, m (Art. 262) have all been given in terms of A_2, I_2, I_3, K with the exception of $p_0m_2-2p_1m_1+p_2m_0$ and the skew invariant (plm).

It is easy to show that

$$p_0m_2-2p_1m_1+p_2m_0=90I_3+5A_2I_2-20I_2K-5A_2K^2+30K^3,$$

whence we can express the square of (plm) by the others. This invariant (plm) is of the ninth degree in the coefficients of each of the quartics as in Art. 222, and the invariant E of the sextic contains it as a factor, for, by the linear relation among p, l, m, n, we have $5\,(lmn) = 72\,(2I_3 - I_2K + K^3)\,(plm)$.

Ex. 10. If the second quartic v of Art. 262 be the Hessian of the first u, the combinant p vanishes identically, and the functional determinant becomes the sextic covariant of the biquadratic u. The identities of that article show that the covariant i of this sextic vanishes identically (*Clebsch*, p. 447).

Ex. 11. If the two quartics u and v of Art. 262 be the derived functions of a quintic, the Jacobian is its Hessian, and their invariant B (Art. 220) vanishes identically. Now in the present notation we have from that article $A + 48B = -40A_2$, $A - 12B = -4P$; thus for $B = 0$, $P = 10A_2$: whence this invariant relation among the quartics is the same as that $K = -\frac{11}{30}A_2$ shall satisfy the quintic of Art. 262. (Stephanos, l.c. p. 81; F. Lindemann, *Math. Ann.* XXI., p. 81.)

Ex. 12. If a quintic and a sextic admit of reduction to the forms

$$A'u^5 + B'v^5 + C'w^5 + D'z^5, \quad Au^6 + Bv^6 + Cw^6 + Dz^6,$$

they satisfy the invariant relation $\begin{vmatrix} a', & b', & a, & b, & c \\ b', & c', & b, & c, & d \\ c', & d', & c, & d, & e \\ d', & e', & d, & e, & f \\ e', & f', & e, & f, & g \end{vmatrix} = 0.$

Ex. 13. In order to reduce two quintics to the forms

$$Au^5 + Bv^5 + Cw^5 + Dz^5, \quad A'u^5 + B'v^5 + C'w^5 + D'z^5,$$

their canonizant $uvwz$ is $\begin{vmatrix} ax + by, & bx + cy, & a'x + b'y, & b'x + c'y \\ bx + cy, & cx + dy, & b'x + c'y, & c'x + d'y \\ cx + dy, & dx + ey, & c'x + d'y, & d'x + e'y \\ dx + ey, & ex + fy, & d'x + e'y, & e'x + f'y \end{vmatrix} = 0.$

Ex. 14. Similarly if a cubic and quintic admit of reduction to the sum of three cubes and three fifth powers of the same quantities, we have $\begin{vmatrix} a', & a, & b, & c \\ b', & b, & c, & d \\ c', & c, & d, & e \\ d', & d, & e, & f \end{vmatrix} = 0.$

Ex. 15. In Hermite's invariant, p. 234, the leading terms in a and f are

$$I = a^7f\,(df - e^2)^5 - af^7\,(ac - b^2)^5 + \&c.$$

Ex. 16. Determine a quintic v such that the result of operating with $\overline{12}^1$ on uv, where u is a given quintic, may vanish identically. We find

$$\begin{vmatrix} x^5, & 5x^4y, & 10x^3y^2, & 10x^2y^3, & 5xy^4, & y^5 \\ d, & -3c, & 3b, & -a, & 0 & 0 \\ e, & -2d, & 0, & 2b, & -a, & 0 \\ f, & e, & -8d, & 8c, & -b, & -a \\ 0, & f, & -2e, & 0, & 2c, & -b \\ 0, & 0, & f, & -3e, & 3d, & -c \end{vmatrix},$$

which is the product of u by its quartinvariant $= -uJ$ of p. 228.

LESSON XIX.

ON THE ORDER OF RESTRICTED SYSTEMS OF EQUATIONS.

267. THE problems discussed in this lesson are purely algebraical, and in the investigation of them I do not make use of any geometrical principles. But I find it convenient to borrow one or two *terms* from geometry, because we can thus avoid circumlocution, and also can more readily see how to extend to quantics in general theorems already known for ternary and quaternary quantics.

We saw (Art. 78)* that if we are given k equations in k independent variables, the number of systems of common values of the variables which can be found to satisfy all the equations, will be equal to the product of the orders of the equations. Now, in the geometry of two and three dimensions respectively, the system of values $x=a$, $y=b$; or $x=a$, $y=b$, $z=c$ denotes a point. I find it convenient therefore to use the word "point" in general instead of "system of values of the variables," so that the theorem already stated may be enunciated: "A system of k equations in k variables of degrees l, m, n, p, q, &c. respectively, represents *lmnpq* &c. *points*," by which we mean that so many "systems of values of the variables" can be found to satisfy all the equations. This number *lmnpq* &c. will be called the *order* of the system of equations.

268. If we have a system of $k-1$ equations in k independent variables, we have not data enough to *determine* any system of common values of the variables, and the system of equations denotes a singly infinite series of "points." Such a system of equations we shall speak of as denoting a *curve*. If with the given system of $k-1$ equations we combine any arbitrary

* If as is usual we employ homogeneous equations, the number of variables will of course be $k+1$.

equation of the first degree, we have then data enough to determine points which will be equal in number to the product of the degrees of the equations. We shall define the *order of a curve* as the number of points which are obtained when, with the equations which denote the curve, we combine an arbitrary equation of the first degree.

When we are given a system of $k-2$ equations, these denote a doubly infinite series of points, since we cannot determine any points unless we are given two other equations. Such a system we shall speak of as denoting a *surface*. If with the system of $k-2$ equations we combine an arbitrary equation of the first degree, we shall have a "curve" whose order is the product of the degrees of the $k-2$ equations. In general, by the *order of a surface*, we mean either the order of the curve obtained by combining with the given equations an equation of the first degree, or, what comes to the same thing, the number of points obtained by combining with the given equations two equations of the first degree.

And so, more generally, if we have any system of fewer than k equations, by the *order of the system* we mean the number of points that are obtained, when with the given equations we combine as many equations of the first degree as are wanting to make the entire number of equations up to k, thus affording data enough to determine systems of values of the variables. It is evident that in the case under consideration, the order of the system is the product of the degrees of the equations which compose it.

269. If we have $k+1$ equations in k independent variables whose degrees are l, m, n, &c., we can eliminate the variables; and we have seen (Arts. 76, 78) that the order in which the coefficients of each equation enter into the resultant, will be equal to the product of the degrees of the remaining equations Taking then, to fix the ideas, the case of four equations: let their orders be l, m, n, r, and let any quantity enter into the coefficients of the equations in the degrees λ, μ, ν, ρ respectively, this quantity will enter into the resultant in the degree

$$\lambda mnr + \mu nrl + \nu rlm + \rho lmn.$$

We shall use the word *order* to denote the degrees l, m, n, r, in which the equations contain the variables which are to be eliminated, and *weight* to denote the degrees λ, μ, ν, ρ in which they contain the quantity not eliminated; and the result just written may be stated, that the weight of the resultant, or the weight of the system, is equal to the sum of the weights of each equation multiplied by the order of the system formed by the remaining equations.

And this is still true, if we break the given system up into partial systems. Thus, the first two equations form a system whose order is lm and weight $\lambda m + \mu l$, and the second two equations a system whose order is nr and weight $\nu r + \rho n$; and the value just given for the weight of the entire system is

$$nr\ (\lambda m + \mu l) + lm\ (\nu r + \rho n),$$

that is, it is the sum of the weights of each component system multiplied by the order of the other. The advantage of so stating the matter will appear presently.

270. What has been hitherto said in this lesson is but a re-statement in other words of principles already laid down in the lesson on Elimination; but my purpose has been to make more intelligible the object of investigations, on which we shall now enter, as to the order and weight of systems of a somewhat different kind. We have seen that k equations in k variables represent $lmnp$ &c. points. But now we may combine with these k equations an additional equation, which is satisfied for some of the points but not for others of them. We have then a system of $k+1$ equations representing points, that is to say, all satisfied by a number of systems of common values of the variables, that number being now, however, generally smaller than the product of the degrees of any k of the equations. Cases are of constant occurrence where a number of points can be expressed in no other way than that here described. A simple geometrical example will suffice. Consider p points in a plane where p is a prime number, and where the points do not lie in a right line, then these points cannot be represented as the complete intersection of any two curves, and if we have any two curves going through the points, their intersection includes

not only these points but others besides. To define the points completely, we must add a third curve going through the p given points, but not through the remaining points of intersection of the first two curves. The points are thus completely defined as the only points common to all three curves. Our object then is, in some important cases where a system of points is defined by more than k equations, to lay down rules for ascertaining the order of the system; that is to say, how many systems of common values satisfy all the equations.

In like manner a system of $k-1$ equations is satisfied by an infinity of common values. But it may happen that we can write down an additional equation satisfied by part of this series of common values, but not by the remaining part. In such a case, the system of $k-1$ equations denotes a complex curve, and it requires the system of k equations to define that part of it for which all the equations are satisfied. It will be the object of this lesson to ascertain the order and weight of what we may call restricted systems; that is to say, where to a number of equations sufficient to define points, curves, &c., is added one or more others which exclude from consideration those values of the variables which satisfy the first set of equations, but do not satisfy the additional equations.

271. The simplest example of such a system is the set of determinants

$$\left\| \begin{matrix} u, & v, & w \\ u', & v', & w' \end{matrix} \right\| = 0,$$

or, at full length,

$$vw' - wv' = 0, \quad wu' - uw' = 0, \quad uv' - vu' = 0.$$

By writing these equations in the form

$$\frac{u}{u'} = \frac{v}{v'} = \frac{w}{w'},$$

it is evident that, in general, values of the variables which satisfy two of the equations must satisfy the third. But there is an exception for the case of values which make either u and u', v and v', or w and $w' = 0$. In any of these cases it is easy to see

that two of the equations will be satisfied, but not the third. And now it is easy to see how to calculate the order of the system common to all three. Let the orders of u and u', of v and v', of w and w', be l, m, n respectively; then the orders of the first two equations are $m+n$, $n+l$, and of the system formed by them is $(m+n)(n+l)$. But in this system will be included values which satisfy both w and w', these values not satisfying the third equation. Excluding then this system, the order of which is n^2, the order of the system common to the three determinants is $mn+nl+lm$.

In like manner, suppose we have a system with three rows and four columns,

$$\left\| \begin{matrix} u, & u', & u'', & u''' \\ v, & v', & v'', & v''' \\ w, & w', & w'', & w''' \end{matrix} \right\| = 0.$$

Let us write at full length the determinants formed by the omission of the third and fourth columns

$$u''(vw'-wv')+v''(wu'-uw')+w''(uv'-vu')=0,$$
$$u'''(vw'-wv')+v'''(wu'-uw')+w'''(uv'-vu')=0,$$

then these two equations are obviously satisfied for all values which satisfy the three $vw'=wv'$, $wu'=uw'$, $uv'=vu'$. But these values will not satisfy the other determinants of the given system. From $(l+m+n)^2$, then, which is the order of the system formed by the two equations written at length, we must subtract $mn+nl+lm$, which has just been found to be the order of the system special to these two equations, and the remainder $l^2+m^2+n^2+mn+nl+lm$ is the order of the system common to all the determinants. Having thus determined the order of a system with three rows and four columns, we can, in like manner, thence derive the order of a system with four rows and five columns. Proceeding thus step by step we arrive by induction at a general formula, for the order of a system with k rows and $(k+1)$ columns.

272. We may consider in succession the cases: 1°, k rows and k columns; 2°, k rows and $(k+1)$ columns; 3°, k rows and $k+2$ columns, and so on. Writing down in each case only the

orders of the several functions, so that $a+\alpha$, $b+\beta$, &c., stand for functions of the orders $a+\alpha$, $b+\beta$, &c. respectively; the case 1° includes the systems

$$|\, a+\alpha \,| = 0, \quad \left| \begin{matrix} a+\alpha, & b+\alpha \\ a+\beta, & b+\beta \end{matrix} \right| = 0, \quad \&c.,$$

the case 2° includes the systems

$$\| \, a+\alpha, \; b+\alpha \, \| = 0, \quad \left\| \begin{matrix} a+\alpha, & b+\alpha, & c+\alpha \\ a+\beta, & b+\beta, & c+\beta \end{matrix} \right\| = 0, \quad \&c.,$$

the case 3° includes the systems

$$\| \, a+\alpha, b+\alpha, c+\alpha \, \| = 0, \quad \left\| \begin{matrix} a+\alpha, & b+\alpha, & c+\alpha, & d+\alpha \\ a+\beta, & b+\beta, & c+\beta, & d+\beta \end{matrix} \right\| = 0, \; \&c.,$$

and so on.

Write in each case

$$C_1 = \Sigma a, \quad C_2 = \Sigma ab, \quad C_3 = \Sigma abc, \; \ldots,$$

so that C_1, C_2, C_3 ... denote the sums of the products without repetitions of the letters a, b, ... (as many of them as belong to the system in question) taken one together, two together, three together, &c., and

$$H_1 = \Sigma \alpha, \quad H_2 = \Sigma \alpha^2 + \Sigma \alpha\beta, \quad H_3 = \Sigma \alpha^3 + \Sigma \alpha^2\beta + \Sigma \alpha\beta\gamma, \; \ldots,$$

or let H_1, H_2, H_3... denote the sums of the homogeneous products, with repetitions of the letters α, β, ... (as many of them as belong to the system in question) taken one together, two together, three together, &c.

Then in the case

1°, the order of the system is $= C_1 + H_1$,

2°, " " $= C_2 + C_1H_1 + H_2$,

3°, " " $= C_3 + C_2H_1 + C_1H_2 + H_3$,

and so on.

Thus in the case 1°, there is only a single equation; and for the several systems written down above, the orders are $a+\alpha$; $a+b+\alpha+\beta$, &c., thus for each system the order is $C_1 + H_1$.

In the case 2°, for the first of the systems written down above, there are two equations of the orders $a+\alpha$, $b+\alpha$ respectively, and the order of the system is $= (a+\alpha)\,(b+\alpha)$, and this is $= ab + (a+b)\,\alpha + \alpha^2$, which is $= C_2 + C_1H_1 + H_2$.

For the second system, viz. the system

$$\left\| \begin{matrix} a+\alpha, & b+\alpha, & c+\alpha \\ a+\beta, & b+\beta, & c+\beta \end{matrix} \right| = 0,$$

which includes the first of those considered in last article, applying to it the reasoning of that article, we have two equations of the orders $a+b+\alpha+\beta$, $a+c+\alpha+\beta$ respectively; the product of these numbers is

$$= a^2 + a(b+c) + bc + (2a+b+c)(\alpha+\beta) + (\alpha^2 + 2\alpha\beta + \beta^2);$$

but we have to subtract from this the product $(a+\alpha)(a+\beta)$, which is $= a^2 + a(\alpha+\beta) + \alpha\beta$; and the order of the system is thus found to be $= ab + ac + bc + (a+b+c)(\alpha+\beta) + \alpha^2 + \beta^2 + \alpha\beta$; which is $= C_2 + C_1 H_1 + H_2$.

The next system is

$$\left\| \begin{matrix} a+\alpha, & b+\alpha, & c+\alpha, & d+\alpha \\ a+\beta, & b+\beta, & c+\beta, & d+\beta \\ a+\gamma, & b+\gamma, & c+\gamma, & d+\gamma \end{matrix} \right| = 0,$$

the order of this is equal to the order of the system

$$\left| \begin{matrix} a+\alpha, & b+\alpha, & c+\alpha \\ a+\beta, & b+\beta, & c+\beta \\ a+\gamma, & b+\gamma, & c+\gamma \end{matrix} \right| = 0, \quad \left| \begin{matrix} a+\alpha, & b+\alpha, & d+\alpha \\ a+\beta, & b+\beta, & d+\beta \\ a+\gamma, & b+\gamma, & d+\gamma \end{matrix} \right| = 0,$$

less the order of the system

$$\left\| \begin{matrix} \alpha+a, & \beta+a, & \gamma+a \\ \alpha+b, & \beta+b, & \gamma+b \end{matrix} \right| = 0,$$

and this is

$$\begin{aligned} &= (a+b+c+\alpha+\beta+\gamma)(a+b+d+\alpha+\beta+\gamma) \\ &\quad - \{a^2+b^2+ab+(a+b)(\alpha+\beta+\gamma)+(\alpha\beta+\alpha\gamma+\beta\gamma)\} \\ &= (a+b+c)(a+b+d)+(2a+2b+c+d)(\alpha+\beta+\gamma)+(\alpha+\beta+\gamma)^2 \\ &\quad - a^2-b^2-ab-(a+b)(\alpha+\beta+\gamma)-\alpha\beta-\alpha\gamma-\beta\gamma \\ &= ab+ac+ad+bc+bd+cd+(a+b+c+d)(\alpha+\beta+\gamma) \\ &\quad + \alpha^2+\beta^2+\gamma^2+\alpha\beta+\alpha\gamma+\beta\gamma, \end{aligned}$$

which is $= C_2 + C_1 H_1 + H_2$,

and similarly for the other systems of the case 2°.

In the case 3°, for the first system we have three equations of the orders $a+\alpha$, $b+\alpha$, $c+\alpha$ respectively; and the order of the system is

$$(a+\alpha)(b+\alpha)(c+\alpha), = abc + (ab+ac+bc)\alpha + (a+b+c)\alpha^2 + \alpha^3$$
$$= C_3 + C_2H_1 + C_1H_2 + H_3;$$

and the result may be verified for the other systems.

273. We may proceed in like manner to calculate the weight of the system of determinants considered in the last article. Beginning again with the simplest case, let us suppose that the system $\left\| \begin{matrix} u, & v, & w \\ u', & v', & w' \end{matrix} \right\|$ is to be combined with one or more other equations and the variables eliminated. Now the result of elimination between $uv'-vu'$, $uw'-wu'$, and any other equations will contain as a factor the resultant of u, u', and the other equations. If we reject this factor we get the same result as if we had eliminated between $uv'-vu'$, $vw'-wv'$ and the other equations, and then rejected the factor got by eliminating between v, v', and the other equations. To illustrate the method employed, let us suppose that u, u'; v, v'; w, w' respectively contain any quantity not eliminated in the degrees λ, μ, ν; and that we are to combine with the determinants of the given system another equation $R=0$, whose order is r, and containing the uneliminated quantity in the degree ρ. This quantity then will enter into the resultant of R, $uv'-vu'$, $uw'-wu'$ in the degree

$$\rho(l+m)(l+n)+r\{(l+m)(\lambda+\nu)+(l+n)(\lambda+\mu)\}.$$

But the resultant of R, u, u', will contain the same quantity in the degree

$$\rho l^2 + 2rl\lambda.$$

When, then, this factor is rejected from the former result, the remainder is

$$\rho(mn+nl+lm)+r\{\lambda(m+n)+\mu(n+l)+\nu(l+m)\}$$

The order then of the system of three determinants is the quantity multiplying ρ, and the weight is the quantity multiplying r.

274. Finding in this way the weight of any system of those considered in Art. 272, the result is that if the orders of the several functions be as written in Art. 272, and if their weights (that is to say, the degrees in which they contain the variable not eliminated) be $a'+\alpha'$, $b'+\alpha'$, &c., $a'+\beta'$, &c., then the formula for the weight is derived from that for the order, by performing on it the operation

$$a'\frac{d}{da}+b'\frac{d}{db}+\&c.+\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\&c.$$

275. If we form the condition that the two equations

$$at^l+bt^{l-1}+ct^{l-2}+\&c.=0,\ a't^m+b't^{m-1}+c't^{m-2}+\&c.=0,$$

should have a common root, we obtain a single equation, namely the resultant of the equations. But if we form the conditions that they should have two common roots, we obtain (Art. 82) not two equations, but a whole system, no doubt equivalent to two conditions, yet such that two equations of the system would not precisely define the conditions in question. Now we may suppose that t is a parameter eliminated, and that a, b, &c. contain variables, and we may propose to investigate the order of the system of conditions in question. Now, Art. 82, these conditions are the determinants of the system

$$\left\|\begin{array}{ccccc} a, & b, & c, & d, & \ldots \\ & a, & b, & c, & \ldots \\ & & a, & b, & \ldots \\ \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots} \\ a', & b', & c', & d', & \ldots \\ & a', & b', & c', & \ldots \\ \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots} \end{array}\right\|,$$

where the first line is repeated $m-1$ times and the second $l-1$ times; there are $l+m-2$ rows and $l+m-1$ columns. The problem is then a particular case of that of Art. 272. We suppose the degrees of the functions introduced to be equidifferent; that is to say, if the degrees of a, a' be λ, μ, we suppose those of b, b' to be $\lambda+\alpha$, $\mu+\alpha$; of c, c' to be $\lambda+2\alpha$, $\mu+2\alpha$, &c. The formula of Art. 272 is, order $= C_2+C_1H_1+H_2$.

To apply it to the present case we may take for the quantities a, b, c, &c. of Art. 272, 0, α, 2α, &c.; and for the quantities α, β, γ, &c., λ, $\lambda-\alpha$, $\lambda-2\alpha$, &c. C_1 is then the sum of $l+m-2$ terms of the series α, 2α, 3α, &c., and is therefore if we write $l+m=k$, $\frac{1}{2}(k-1)(k-2)\alpha$. In the same case C_2 is the sum of products in pairs of these quantities, and is therefore

$$= \frac{(k-3)(k-2)(k-1)(3k-4)}{1.2.3.4.}\alpha^2.$$

Again H_1 is the sum of $m-1$ terms of the series $\lambda, \lambda-\alpha, \lambda-2\alpha$, &c., and of $l-1$ terms of the series μ, $\mu-\alpha$, $\mu-2\alpha$, &c. We have then

$$H_1 = (m-1)\lambda + (l-1)\mu - \tfrac{1}{2}\alpha\{(m-1)(m-2)+(l-1)(l-2)\}.$$

Moreover $H_2 = \frac{1}{2}(H_1^2 + S_2)$, where S_2, the sum of the squares of the same $m-1$ and $l-1$ terms is

$$= (m-1)\lambda^2 + (l-1)\mu^2 - \lambda\alpha(m-1)(m-2) - \mu\alpha(l-1)(l-2)$$
$$+ \alpha^2\left\{\frac{(m-1)(m-2)(2m-3)}{1.2.3} + \frac{(l-1)(l-2)(2l-3)}{1.2.3}\right\}.$$

Collecting all the terms, the order of the required system is found to be

$$\tfrac{1}{2}m(m-1)\lambda^2 + \tfrac{1}{2}l(l-1)\mu^2 + (l-1)(m-1)\lambda\mu$$
$$+ \tfrac{1}{2}m(m-1)(2l-1)\lambda\alpha + \tfrac{1}{2}l(l-1)(2m-1)\mu\alpha$$
$$+ \tfrac{1}{2}lm(l-1)(m-1)\alpha^2.$$

If the two equations considered are of the same degree, that is to say, if $l=m$, we may write $\lambda+\mu=p$, $\lambda\mu=q$, and the order becomes

$$\tfrac{1}{2}m(m-1)(p+m\alpha)\{p+(m-1)\alpha\} - (m-1)q.$$

If all the functions a, b, &c. are of the first degree, writing $\lambda=\mu=1$, and $\alpha=0$ in the preceding formula, the order is found to be $\frac{1}{2}(l+m-1)(l+m-2)$.

276. If the degrees in which the uneliminated variables occur in any terms be denoted by the accented letters corresponding to those which express their degrees in the variables to be eliminated, then the formula for the weight of the system is

obtained from that for the order by performing on it the operation $\lambda' \frac{d}{d\lambda} + \mu' \frac{d}{d\mu} + \alpha' \frac{d}{d\alpha}$. In other words the weight is

$$m(m-1)\lambda\lambda' + l(l-1)\mu\mu' + (l-1)(m-1)(\lambda\mu' + \mu\lambda')$$
$$+ \tfrac{1}{2}m(m-1)(2l-1)(\lambda\alpha' + \alpha\lambda') + \tfrac{1}{2}l(l-1)(2m-1)(\mu\alpha' + \alpha\mu')$$
$$+ lm(l-1)(m-1)\alpha\alpha'.$$

277. The next system we discuss is that formed by the system of conditions that the three equations

$$at^l + bt^{l-1} + \&c. = 0, \quad a't^m + b't^{m-1} + \&c. = 0, \quad a''t^n + b''t^{n-1} + \&c. = 0,$$

may have a common factor. The system may be expressed by the three equations obtained by eliminating t in turn between every pair of these equations, a system equivalent to two conditions. The order of the system may be found by eliminating from the equations the variables which enter implicitly into a, b, c, &c., when the order of the resulting equation in t determines the order of the system.

Let us suppose that a, a', a'' are homogeneous functions in x, y, z of the degrees λ, μ, ν respectively; that b, b', b'' are of the degrees $\lambda - 1$, $\mu - 1$, $\nu - 1$, &c., and if we take the reciprocal of t as a fourth variable, the equations are of the orders respectively λ, μ, ν, forming a system of the order $\lambda\mu\nu$. But the system of values $x = 0, y = 0, z = 0$ is a multiple point in the three equations of the orders $\lambda - l$, $\mu - m$, $\nu - n$ respectively. The order then is to be reduced by $(\lambda - l)(\mu - m)(\nu - n)$. It is therefore

$$l\mu\nu + m\nu\lambda + n\lambda\mu - \lambda mn - \mu nl - \nu lm + lmn.$$

This then is the order of the system we are investigating. If the orders of b, b', c, c', &c. had been $\lambda + \alpha$, $\mu + \alpha$, $\lambda + 2\alpha$, $\mu + 2\alpha$, &c., then the order of the system would have been

$$l\mu\nu + m\nu\lambda + n\lambda\mu + \alpha(mn\lambda + nl\mu + lm\nu) + \alpha^2 lmn.$$

The weight is found by operating on this with $\lambda' \frac{d}{d\lambda}$ + &c., and is

$$l(\mu\nu' + \nu\mu') + m(\nu\lambda' + \lambda\nu') + n(\lambda\mu' + \mu\lambda') + mn(\alpha\lambda' + \lambda\alpha')$$
$$+ nl(\alpha\mu' + \mu\alpha') + lm(\alpha\nu' + \nu\alpha') + 2lmn\alpha\alpha'.$$

278. It is a particular case of the preceding to find the order and weight of the system of conditions that an equation $at^n + bt^{n-1} + \&c. = 0$ may have three equal roots; because these conditions are found by expressing that the three second differentials may have a common factor. Writing in the preceding for l, m; for n, $n-2$; for μ, $\lambda + \alpha$; and for ν, $\lambda + 2\alpha$; we find, for the order of the system,

$$3(n-2)\lambda(\lambda + n\alpha) + n(n-1)(n-2)\alpha^2;$$

and in like manner for its weight

$$6(n-2)\lambda\lambda' + 3n(n-2)(\lambda\alpha' + \alpha\lambda') + 2n(n-1)(n-2)\alpha\alpha'.$$

Again, to find the order and weight of the system of conditions that the same equation may have two distinct pairs of equal roots, we form first, by Art. 272, the order and weight of the system of conditions that the two first differentials $at^{n-1} + \&c.$, $bt^{n-1} + \&c.$ may have two common factors. We subtract then the order and weight of the system found in the first part of this article. The result is that the order is

$$2(n-2)(n-3)\lambda(\lambda + n\alpha) + \tfrac{1}{2}n(n-1)(n-2)(n-3)\alpha^2,$$

and the weight is

$$4(n-2)(n-3)\lambda\lambda' + 2n(n-2)(n-3)(\lambda\alpha' + \alpha\lambda') + n(n-1)(n-2)(n-3)\alpha\alpha'.$$

Before proceeding further in investigating the order of other systems, it is necessary to discuss a different problem, and I commence by explaining the use of one or two other terms which I borrow from geometry.

279. *Intersection of quantics having common curves.* Two systems of quantics are said to *intersect* if they have one or more "points" common, that is to say, if they are both capable of being satisfied by the same system of values of the variables. A "surface" is said to *contain* a "curve" if every system of values which satisfies the $k-1$ equations constituting the curve, satisfies also the $k-2$ equations constituting the surface. Thus, in the case of four variables, three equations $U=0$, $V=0$, $W=0$ constitute a curve, and the two equations $U=0$ $V=0$ constitute a surface which evidently contains that curve.

Now a system of k quantics in k variables, in general, as we have seen, intersect in a definite number of points, that number being the product of the orders of the quantics. But it may happen that they may have an infinity of points common, these points forming a "curve" in the sense in which we have already defined that word. Besides that curve they will have ordinarily a finite number of points common, which it is our object now to determine. Let us take, for example, to fix the ideas, the case of four independent variables; and suppose that we have four equations of the form

$$\begin{aligned} U &= Au + Bv + Cw = 0, \\ V &= A'u + B'v + C'w = 0, \\ W &= A''u + B''v + C''w = 0, \\ Z &= A'''u + B'''v + C'''w = 0. \end{aligned}$$

We suppose the degrees of U, V, W, Z to be l, m, n, p; of u, v, w to be λ, μ, ν; and A, B; A', B', &c. are therefore functions of the degrees $l-\lambda, l-\mu$; $m-\lambda$, $m-\mu$, &c. Now, evidently, these equations will be all satisfied by every system of values which make $u=0, v=0, w=0$; and these equations not being sufficient to determine "points," will be satisfied by an infinity of values of variables. In other words, the four quantics U, V, W, Z have a common curve uvw. And yet U, V, W, Z may be satisfied by a number of values which do not make u, v, w all $=0$. It is our object to determine this latter number; and our problem is, When a system of quantics has a common curve, to find how many of their $lmnp$, &c. points of intersection are absorbed by that curve, and in how many points they intersect not on that curve.

280. Let us first consider the curve formed by $k-1$ of the quantics; for instance, in the example we have chosen for illustration, the curve UVW. Now evidently a portion of this curve is the curve uvw, but there are besides an infinity of points satisfying UVW which do not satisfy u, v, w. We speak then of the curve UVW, as a complex curve consisting of the curve uvw and a complementary curve. Now the order of a complex curve is always equal to the sum of the orders of its

components. For, by definition, the order of the complex curve UVW is the number of points obtained by combining with the equations of the system an additional one of the first degree: that order being in the present case lmn. And evidently, since of those lmn points $\lambda\mu\nu$ lie on the curve uvw, there must be $lmn - \lambda\mu\nu$ on the complementary curve.

The two curves intersect in points whose number i is easily obtained. For evidently all points which satisfy the three equations

$$Au + Bv + Cw = 0, \quad A'u + B'v + C'w = 0, \quad A''u + B''v + C''w = 0,$$

and which do not satisfy u, v, w; must satisfy the determinant

$$\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix} = 0,$$

the degree of which is $l + m + n - \lambda - \mu - \nu$. The intersection of this new quantic with uvw gives all the points in which uvw meets the complementary curve. We have therefore

$$i = \lambda\mu\nu\,(l + m + n - \lambda - \mu - \nu).$$

281. To find now the number of points common to $UVWZ$, we have to consider the points in which the curve UVW meets Z; and it is required to find how many of these are not on the curve uvw. But since uvw is itself a part of the curve UVW, it is evident that the points required are contained among the $p\,(lmn - \lambda\mu\nu)$ points in which the complementary curve meets Z. And from these points must be excluded the i points in which the complementary curve meets uvw. Using then the value given in the last article for i, we find, for the number which we seek to determine,

$$lmnp - \lambda\mu\nu\,(l + m + n + p) + \lambda\mu\nu\,(\lambda + \mu + \nu).$$

We shall state this result thus, that if k quantics of orders l, m, n, p, &c. have common a curve of order α; then the number of points which they will have common in addition to this curve is less than the product of the orders of the quantics by $\alpha\,(l + m + n + \&c.) - \beta$, where β is a constant depending only on the nature of the curve and not involving the orders

of the quantics. We shall call this constant the *rank* of the curve. We have seen that when the curve is given as the intersection of quantics u, v, w, the order is $\lambda\mu\nu$ and the rank $\lambda\mu\nu\,(\lambda+\mu+\nu)$.

We saw, in the last article, that if the intersection UVW consists of two complementary curves whose orders are α, α', and whose ranks are β, β', the number of points in which the two curves intersect is $\alpha\,(l+m+n)-\beta$; and by parity of reasoning it is $\alpha'\,(l+m+n)-\beta'$; Hence the orders and ranks of the two complementary curves are connected by the equations $\alpha+\alpha'=lmn$, $\beta-\beta'=(\alpha-\alpha')(l+m+n)$.

282. Next, let us consider the case where the quantics have common two or more distinct curves uvw, $u'v'w'$, &c. Let the intersection for instance of UVW consist of the two curves uvw, $u'v'w'$, and of a complementary curve α''; then, in the first place, the order of α'' is evidently $lmn-\lambda\mu\nu-\lambda'\mu'\nu'$. Secondly, we have seen that uvw meets the remaining intersection of UVW in points whose number is

$$\lambda\mu\nu\,(l+m+n-\lambda-\mu-\nu).$$

If then i of these lie on $u'v'w'$ (that is to say, if uvw, $u'v'w'$ intersect in i points) there must be on the complementary curve α''

$$\lambda\mu\nu\,(l+m+n-\lambda-\mu-\nu)-i.$$

And in like manner α'' meets $u'v'w'$ in

$$\lambda'\mu'\nu'\,(l+m+n-\lambda'-\mu'-\nu')-i \text{ points.}$$

As before, then, the number of points on neither curve in which α'' meets any other quantic Z is

$$(lmn-\lambda\mu\nu-\lambda'\mu'\nu')\,p-\lambda\mu\nu\,(l+m+n-\lambda-\mu-\nu)$$
$$-\lambda'\mu'\nu'\,(l+m+n-\lambda'-\mu'-\nu')+2i,$$

or $$lmnp-(\lambda\mu\nu+\lambda'\mu'\nu')\,(l+m+n+p)+\lambda\mu\nu\,(\lambda+\mu+\nu)$$
$$+\lambda'\mu'\nu'\,(\lambda'+\mu'+\nu')+2i.$$

Thus, then, the diminution from the number $lmnp$ effected by a complex curve is equal to the sum of the diminutions effected by the simple curves less double the number of their points of intersection. The same holds no matter how many

be the curves common to the quantics; and we may say that when a complex curve consists of several simple curves the order of the complex is equal to the sum of the orders of its components; and the rank of the complex is equal to the sum of their ranks increased by double the number of points common to every pair of curves.

283. We give, as an illustration of the application of these principles, the problem to determine how many surfaces of the second degree can be described through five points to touch four planes. Let S, T, U, V, W be five surfaces passing through the five points, then any other will be of the form $\alpha S + \beta T + \gamma U + \delta V + \varepsilon W$; and the condition that this should touch a plane will be a cubic function of the five quantities α, β, γ, δ, ε. We are given four such equations, and it is required to find how many systems of values can be got to satisfy them all. If the four equations had no common "curves" the number of their common "points" would be 3^4 or 81. But the existence of common curves may be seen in this way: The condition that a surface of the second order should touch a plane vanishes identically when the surface consists of two planes. Let us take then for S and T two pairs of planes passing through the five given points, $S = (123)(145)$, $T = (123)(245)$; then evidently, the condition that $\alpha S + \beta T + \gamma U + \delta V + \varepsilon W$ should touch any plane whatever, must be satisfied by the supposition $\gamma = 0$, $\delta = 0$, $\varepsilon = 0$. This "curve," then, which is of the first degree, will be common to all four quantics. And, if we call this the line (123) (45), it is evident, by parity of reasoning, that the quantics have common ten such lines (124) (35), &c. Now if, as before, we take S as the system of two planes (123) (145), $T = (123)(245)$, and take $U = (145)(234)$; then, while the line (123) (45) is denoted by $\gamma = 0$, $\delta = 0$, $\varepsilon = 0$, the line (145) (23) is denoted by $\beta = 0$, $\delta = 0$, $\varepsilon = 0$; and these two lines intersect, being both satisfied by the common values $\beta = 0$, $\gamma = 0$, $\delta = 0$, $\varepsilon = 0$. And, in like manner, (123) (45) is intersected by (245) (13), (345) (12). Thus, then, the ten lines have fifteen points of mutual intersection. The rank of a single curve of the first degree being got by making $\lambda = \mu = \nu = 1$ in

the formula $\lambda\mu\nu\,(\lambda+\mu+\nu)$ is three. Hence the rank of the entire system is ten times three increased by twice fifteen or is 60. And the number of points which satisfy the four quantics is $81-10\,(3+3+3+3)+60$ or is 21.

284. We have shown, Art. 272, how to determine the order of a system of determinants, the number of rows and columns in whose matrix differ by one. We shall now show how, in the last mentioned case, to determine the rank of the curve. Commence, as before, with the simple case

$$\left\| \begin{matrix} u, & v, & w \\ u', & v', & w' \end{matrix} \right\|,$$

and we see that the intersection of $uv'-vu'$, $uw'-wu'$ is a complex curve, consisting of the curve uu' and of the curve with which we are concerned, and knowing the order and rank of uu', we find the order and rank of the other curve. Representing as before the orders of the several terms by

$$\left| \begin{matrix} \alpha+\alpha, & b+\alpha, & c+\alpha \\ a+\beta, & b+\beta, & c+\beta \end{matrix} \right|,$$

we thus obtain

$$\text{Rank} = \text{rank of } (uv'-vu', uw'-wu') - \text{rank of } (u, u')$$
$$-\text{twice number of intersections of the two curves,}$$

and this is

$$=(a+b+\alpha+\beta)\,(a+c+\alpha+\beta)\,(2a+b+c+2\alpha+2\beta)$$
$$-(a+\alpha)\,(a+\beta)\,(2a+\alpha+\beta)$$
$$-2\,(a+\alpha)\,(a+\beta)\,(b+c+\alpha+\beta),$$

or, introducing the former notation (see Art. 272),

$$C_1=a+b+c,\ \&c.,\quad H_1=\alpha+\beta,\ \&c.,$$

this is $$=(a+b+H_1)\,(a+c+H_1)\,(a+C_1+2H_1)$$
$$-(a^2+aH_1+H_1^2-H_2)\,(2C_1+3H_1);$$

or, what is the same thing,

$$=\{a^2+C_2+(a+C_1)\,H_1+H_1^2\}\,(a+C_1+2H_1)$$
$$-(a^2+aH_1+H_1^2-H_2)\,(2C_1+3H_1),$$

which is easily found to be

$$\begin{aligned} = \; & C_3 + C_1 C_2 \\ & + H_1 (C_1^2 + 2C_2) \\ & + C_1 (H_1^2 + 2H_2) \\ & - H_1^3 + 3H_1 H_2, \end{aligned}$$

or attending to the relation $H_3 - 2H_1H_2 + H_1^3 = 0$ which exists in the case of two equations (α, β), this is

$$\begin{aligned} = \; & C_3 + C_1 C_2 \\ & + H_1 (C_1^2 + 2C_2) \\ & + C_1 (H_1^2 + 2H_2) \\ & + H_3 + H_1 H_2; \end{aligned}$$

or, finally, the rank is

$$= (C_3 + C_2 H_1 + C_1 H_2 + H_3) + (C_2 + C_1 H_1 + H_2)(C_1 + H_1),$$

and passing successively to the cases of four columns and three rows, five columns and four rows, &c., it may be shown that C_1, C_2, C_3 referring to the series of numbers, a, b, c, &c., and H_1, H_2, H_3 to the series of numbers, α, β, &c., the foregoing expression for the rank holds good for the system in which the number of the rows and columns differ by one.

285. The formula of the last article may be applied to calculate the order of the system of conditions, that the equations $at^m + $ &c., $a't^n + $ &c. may have three common roots. The conditions are formed by a system of determinants, the matrix for which is formed as in Art. 275; save that the line a, b, c is repeated $n-2$ times, and the line a', b', c', $m-2$ times. The matrix consists of $m+n-2$ columns and $m+n-4$ rows. The order of the system then calculated by the last article is found to be

$$\begin{aligned} & \frac{n(n-1)(n-2)}{1.2.3}\lambda^3 + \frac{m(m-1)(m-2)}{1.2.3}\mu^3 + \tfrac{1}{2}(n-1)(n-2)(m-2)\lambda^2\mu \\ & + \tfrac{1}{2}(m-1)(m-2)(n-2)\lambda\mu^2 + \tfrac{1}{2}(m-1)n(n-1)(n-2)\lambda^2\alpha \\ & + \tfrac{1}{2}(n-1)m(m-1)(m-2)\mu^2\alpha + \tfrac{1}{2}(m-2)(n-2)\{m(n-1)+n(m-1)\}\lambda\mu\alpha \\ & + \{\tfrac{1}{2}n(n-1)(n-2)m(m-2) + \tfrac{1}{3}n(n-1)(n-2)\}\alpha^2\lambda \\ & + \{\tfrac{1}{2}m(m-1)(m-2)n(n-2) + \tfrac{1}{3}n(m-1)(m-2)\}\alpha^2\mu \\ & + \tfrac{1}{6}m(m-1)(m-2)n(n-1)(n-2)\alpha^3. \end{aligned}$$

In the case where we have $\alpha = 0$, $\lambda = \mu = 1$, this reduces to

$$\tfrac{1}{6}(m+n-2)(m+n-3)(m+n-4).$$

The weight of the system, found by the same process as before is

$$\tfrac{1}{2}n(n-1)(n-2)\lambda^2\lambda' + \tfrac{1}{2}m(m-1)(m-2)\mu^2\mu'$$
$$+ (n-1)(n-2)(m-2)(\lambda\mu\lambda' + \tfrac{1}{2}\lambda^2\mu') + (m-1)(m-2)(n-2)(\lambda\mu\mu' + \tfrac{1}{2}\mu^2\lambda')$$
$$+ (m-1)n(n-1)(n-2)(\lambda\lambda'\alpha + \tfrac{1}{2}\lambda^2\alpha')$$
$$+ (n-1)m(m-1)(m-2)(\mu\mu'\alpha + \tfrac{1}{2}\mu^2\alpha')$$
$$+ \tfrac{1}{2}(m-2)(n-2)\{2mn - m - n\}(\lambda\mu'\alpha + \lambda'\mu\alpha + \lambda\mu\alpha')$$
$$+ \{\tfrac{1}{2}n(n-1)(n-2)m(m-2) + \tfrac{1}{3}n(n-1)(n-2)\}(\alpha^2\lambda' + 2\alpha\alpha'\lambda)$$
$$+ \{\tfrac{1}{2}m(m-1)(m-2)n(n-2) + \tfrac{1}{3}m(m-1)(m-2)\}(\alpha^2\mu' + 2\alpha\alpha'\mu)$$
$$+ \tfrac{1}{2}m(m-1)(m-2)n(n-1)(n-2)\alpha^2\alpha'.$$

286. The next problem we investigate is when a system of quantics have a "surface" common, to find how many of their points of intersection are absorbed by the common surface. We mean by the *order* and *rank of a surface*, the order and rank of the curve which is the section of the surface by any quantic of the first degree. Thus, consider the case of five independent variables, then a system of three equations constitutes a surface, and if their orders be λ, μ, ν, the order of the surface will be $\lambda\mu\nu$, and its rank $\lambda\mu\nu(\lambda+\mu+\nu)$; these being the order and rank of the curve got by uniting with the given equations an additional one of the first degree.

Now, first let $k-1$ quantics have a surface in common, whose order and rank are α, β; they will also in general have common besides a complementary curve whose order is readily found. Thus if $k=5$, joining with the given quantics another of the first degree, we then have a system of 5 quantics, having a curve common, and therefore by Art. 281 intersecting in $lmnr - \alpha(l+m+n+r) + \beta$ points besides. But these are the points in which the quantic of the first degree meets the complementary curve, and therefore this is the order of that curve.

287. Next let us investigate the number of points in which the surface and complementary curve intersect each other. Let U, V, W, Y (being as above of the orders l, m, n, r respectively) be respectively of the forms

$$\begin{aligned} Au + Bv + Cw &= 0, \\ A'u + B'v + C'w &= 0, \\ A''u + B''v + C''w &= 0, \\ A'''u + B'''v + C'''w &= 0, \end{aligned}$$

where u, v, w are of the orders λ, μ, ν respectively.

Then the points common to U, V, W, Y which do not make $u, v, w = 0$, will satisfy the system of determinants

$$\left\| \begin{matrix} A, & A', & A'', & A''' \\ B, & B', & B'', & B''' \\ C, & C', & C'', & C''' \end{matrix} \right\| = 0.$$

But since A is of the order $l - \lambda$, B of the order $l - \mu$, A' of the order $m - \lambda$, &c., it follows (Art. 272) that the order of the set of determinants is

$$\begin{aligned} &(lm + ln + lr + mn + mr + nr) \\ &- (l + m + n + r)(\lambda + \mu + \nu) \\ &+ (\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \lambda\nu + \mu\nu). \end{aligned}$$

If now we combine this system of determinants (equivalent to two conditions), with the $k - 3$ conditions which constitute the surface, we determine the points common to the surface and complementary curve. And their number is the order of the system of determinants, multiplied by $\lambda\mu\nu$. Writing then α and β for the order and rank of the surface $\lambda\mu\nu$, $\lambda\mu\nu(\lambda + \mu + \nu)$, and denoting by γ the new characteristic

$$\lambda\mu\nu(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \mu\nu + \nu\lambda),$$

which we may call the *class* of the surface, we find

$$\begin{aligned} i = &\alpha(lm + ln + lr + mn + mr + nr) \\ &- \beta(l + m + n + r) \\ &+ \gamma. \end{aligned}$$

288. If then we have an additional quantic Z also containing the given surface, and if it be required to find how many points not on the surface are common to all 5 quantics,

these will be evidently the points of intersection of the complementary curve with Z, less the number of points of intersection of the complementary curve with the surface. If then l, m, n, r, s be the orders of the quantics, the number sought will be got by subtracting from

$$s\{lmnr - \alpha(l+m+n+r)+\beta\},$$

the number

$$\alpha(lm+ln+mn+lr+mr+nr)-\beta(l+m+n+r)+\gamma.$$

And the difference is

$$\begin{aligned} &lmnrs \\ &-\alpha(lm+\ldots+rs) \\ &+\beta(l+m+n+r+s)-\gamma, \end{aligned}$$

which is the formula required.

289. Next let us consider the case ($k=5$) where a system of quantics have common not only a surface, whose characteristics are α, β, γ, but also a curve, whose characteristics are α', β', intersecting the surface in i points. As before, consider first 4 of the quantics. Their intersection we have seen consists of the surface and of a complementary curve, whose order is

$$lmnr-\alpha(l+m+n+r)+\beta.$$

And if the complementary curve be itself complex, consisting in part of the curve α', and also of another curve, whose order is α'', we have evidently

$$\alpha''=lmnr-\alpha(l+m+n+r)+\beta-\alpha'.$$

The points therefore which we desire to determine are got by subtracting from the $s\alpha''$ points of intersection of the curve α'' with the remaining one of the given system of 5 quantics, $\delta+\delta'$ where δ is the number of points in which the curve α'' meets the surface (α, β, γ), and δ' is the number of points where it meets the curve. But we know δ, since we know, by Art. 287, the number of points where the surface is met by the entire curve complementary to it; and therefore have

$$\delta+i=\alpha(lm+ln+\&c.)-\beta(l+m+n+r)+\gamma;$$

and we know δ', knowing, by Art. 280, the number of points

in which the curve α' is met by the entire curve complementary to it, and therefore have

$$\delta' + 2i = \alpha'(l + m + n + r) - \beta'.$$

Substituting the values thence derived for δ and δ' in $s\alpha'' - \delta - \delta'$, we get

$$\begin{aligned} & lmnrs \\ & - \alpha (lm + \ldots + rs) \\ & + \beta (l + m + n + r + s) \\ & - \gamma \\ & - \alpha' (l + m + n + r + s) \\ & + \beta' \\ & + 3i. \end{aligned}$$

In other words, the diminution from the number $lmnrs$ produced by curve and surface together is equal to the sum of their separate diminutions lessened by three times the number of their common points.

290. This result may be confirmed by supposing one of the quantics to be a complex one $Z'Z''$, where Z' contains the common surface, and Z'' the common curve; and the degrees of Z', Z'' are s', s''. Then the quantics U, V, W, Y, Z', by Art. 288, have common points not on the common surface

$$lmnrs' - \alpha\{s'(l + m + \&c.) + lm + mn + \&c.\} + \beta(s' + l + m + \&c.) - \gamma.$$

But among these will be reckoned the $\alpha's'$ points in which the common curve meets Z', deducting however the i points common to the curve and surface. To find then the number of points $UVWYZ'$ which lie on neither curve nor surface, we must deduct from the number last written $\alpha's' - i$.

Consider now the intersections of U, V, W, Y, Z''; these are a system of quantics having common two curves intersecting in i points; viz. the given curve α', and the curve of intersection of the common surface by Z'', whose order will be $\alpha s''$, and whose rank will be $\alpha s''(\lambda + \mu + \nu + s'')$. The number of points $UVWYZ''$ which lie on neither curve nor surface will be

$$lmnrs'' - (\alpha' + \alpha s'')(l + m + n + r + s'') + \beta' + \alpha s''(\lambda + \mu + \nu + s'') + 2i.$$

Adding, and writing s for $s' + s''$, we get

$$lmnrs - \alpha (lm + \ldots + rs) + \&c.,$$

as in the last article.

291. We next suppose the quantics to have common two surfaces having i points of intersection. The method would be the same if there were several surfaces. Let the last quantic be a complex one, consisting of Z' which passes through the first surface and Z'' which passes through the second. Then the system U, V, W, Y, Z', have the common surface $\lambda\mu\nu$ and the curve $\lambda'\mu'\nu's'$, which have i points common, and the number of points of intersection, not lying on either surface, is thus

$$lmnrs' - \lambda\mu\nu \{(l+m+n+r)\, s' + lm + \&c.\} + \beta\, (l+m+n+r+s') - \gamma - \lambda'\mu'\nu's'\, (l+m+n+r+s') + \lambda'\mu'\nu's'\, (\lambda'+\mu'+\nu'+s') + 3i.$$

In like manner for the system U, V, W, Y, Z'', the number of points of intersection not lying on either surface, is

$$lmnrs'' - \lambda\mu\nu s''\, (l+m+n+r+s'') + \lambda\mu\nu s''\, (\lambda+\mu+\nu+s'') - \lambda'\mu'\nu' \{(l+m+n+r)\, s'' + lm + \&c.\} + \beta'\, (l+m+n+r+s'') + 3i.$$

Adding these, we have for the whole number of points of intersection

$$lmnr\, (s'+s'') - (\lambda\mu\nu + \lambda'\mu'\nu') \{(l+m+n+r)\, (s'+s'') + lm + \&c.\} + (\beta+\beta')\, (l+m+n+r+s'+s'') - \gamma - \gamma' + 6i.$$

In other words, the combined effect of the two surfaces is equal to the sum of the effects of the surfaces separately considered, diminished by six times the number of their common points. When there are only four variables, two surfaces always *must* have common points of intersection.

292. Lastly, let the two surfaces have a common curve whose order and rank are α'', β''. Proceeding, as in the last article, we find that the system $UVWYZ'$ have common indeed the surface $\lambda\mu\nu$, and the curve $\lambda'\mu'\nu's'$. But since this curve is a complex one, consisting in part of the curve α'', β'' which lies on $\lambda\mu\nu$, we are only to take into account the complementary curve which, by Art. 282, has for its order

$\lambda'\mu'\nu's' - \alpha''$, while its rank is

$$\beta'' + (\lambda'\mu'\nu's' - 2\alpha'')(\lambda' + \mu' + \nu' + s');$$

and the complementary curve intersects $\alpha''\beta''$ in

$$\alpha''(\lambda' + \mu' + \nu' + s') - \beta'' \text{ points.}$$

The number of intersections is therefore

$$lmnrs' - \lambda\mu\nu\{(l+m+n+r)s' + lm + \&c.\} + \beta(l+m+n+r+s') - \gamma$$
$$- (\lambda'\mu'\nu's' - \alpha'')(l+m+n+r+s')$$
$$+ \beta'' + (\lambda'\mu'\nu's' - 2\alpha'')(\lambda' + \mu' + \nu' + s') + 3\alpha''(\lambda' + \mu' + \nu' + s') - 3\beta''.$$

Similarly the intersections for $UVWYZ''$ are

$$lmnrs'' - \lambda'\mu'\nu'\{(l+m+n+r)s'' + lm + \&c.\} + \beta'(l+m+n+r+s'') - \gamma'$$
$$- (\lambda\mu\nu s'' - \alpha'')(l+m+n+r+s'')$$
$$+ \beta'' + (\lambda\mu\nu s'' - 2\alpha'')(\lambda + \mu + \nu + s'') + 3\alpha''(\lambda + \mu + \nu + s'') - 3\beta''.$$

Adding, we have

$$lmnr(s' + s'') - (\lambda\mu\nu + \lambda'\mu'\nu')\{(l+m+n+r)(s' + s'') + lm + \&c.\}$$
$$+ (\beta + \beta' + 2\alpha'')(l+m+n+r+s'+s'')$$
$$- \gamma - \gamma' + \alpha''(\lambda + \mu + \nu + \lambda' + \mu' + \nu') - 4\beta''.$$

In other words, the diminution is obtained by regarding the two surfaces as making up a complex surface, whose order is the sum of their orders, whose rank is the sum of their ranks increased by twice the order of the common curve, and whose class is the sum of their classes increased by four times the rank of the common curve and diminished by $\alpha''(\lambda + \mu + \nu + \lambda' + \mu' + \nu')$.

We must leave untouched some other cases which ought to be discussed in order to complete the subject; in particular the case where the surfaces touch in points or along a curve.

293. We come now to the problem of finding the order of the system of conditions that three ternary quantics should have two common points. The method followed is the same as that given by Prof. Cayley for eliminating between three homogeneous equations in three variables, and which we have explained (Art. 94). Let the three equations be of the degrees l, m, n. Multiply the first by all the terms x^{m+n-3}, yx^{m+n-4}, &c.

of an equation of the degree $m+n-3$, the second in like manner by all the terms of an equation of the degree $n+l-3$, and the third by all the terms of an equation of the degree $l+m-3$. We have thus in all

$$\tfrac{1}{2}(m+n-1)(m+n-2)+\tfrac{1}{2}(n+l-1)(n+l-2)+\tfrac{1}{2}(l+m-1)(l+m-2)$$

equations of the degree $l+m+n-3$, from which we are to eliminate the $\frac{1}{2}(l+m+n-1)(l+m+n-2)$ terms $x^{l+m+n-3}$, &c. But, as it has been shewn in the place referred to, the equations we use are not independent, but are connected by

$$\tfrac{1}{2}(l-1)(l-2)+\tfrac{1}{2}(m-1)(m-2)+\tfrac{1}{2}(n-1)(n-2).$$

relations. Subtracting then the number of relations, the number of independent equations is found to be one less than the number of quantities to be eliminated; and we have a matrix in which the number of columns is one more than the number of rows, the case considered in Art. 272. But, as was shewn, Art. 93, when we are given a number of equations connected by relations, the determinants formed by taking a sufficient number of the equations, require to be reduced by dividing out extraneous factors, these factors being determinants formed with the coefficients of the equations of relation. If then, in the present case, we took a sufficient number of the equations and determined the order by the rule of Art. 272, our result would require to be reduced by a number which we proceed to determine.

294. Let us commence with the simplest case where we have k equations in k variables, the equations being connected by a single relation. To fix the ideas we write down the system with three rows

$$\left\| \begin{array}{ccc|c} a, & b, & c & \lambda \\ a', & b', & c' & \lambda' \\ a'', & b'', & c'' & \lambda'' \end{array} \right\|,$$

where we mean to imply that the quantities involved are connected by the relations

$$\lambda a+\lambda' a'+\lambda'' a''=0,\ \lambda b+\lambda' b'+\lambda'' b''=0,\ \lambda c+\lambda' c'+\lambda'' c''=0.$$

We also suppose that λa, $\lambda' a'$, $\lambda'' a''$ are of the same order, so that the orders $\lambda+a=\lambda'+a'=\lambda''+a''$. Now let us, in the first

place, suppose that the two first equations are in the simplest form, and that $\lambda'' = -1$. The true order then is that determined from the first two equations; that is to say, if we indicate the orders, as in Art. 272, $C_2 + C_1(\alpha+\beta) + \alpha^2 + \beta^2 + \alpha\beta$. Now suppose that we had omitted the first row, the order deduced from the second and third would be $C_2 + C_1(\beta+\gamma) + \beta^2 + \gamma^2 + \beta\gamma$; which we see, in order to give the true order, requires to be reduced by $(\gamma-\alpha)(C_1 + \alpha + \beta + \gamma)$; in other words, by the order of λ multiplied by the order of the determinant obtained from the three equations. And the general rule to which we are thus led is: Leave out one of the rows and determine the order of the remaining system by the rule of Art. 272; from the number so found, subtract the order of the determinant formed from all the equations, multiplied by the order of the term in the relation column belonging to the omitted row. It is easy to verify, that we are thus led to the same result whatever be the omitted row. Thus

$$C_2 + C_1(\alpha+\beta) + \alpha^2 + \beta^2 + \alpha\beta - \lambda''(C_1 + \alpha + \beta + \gamma)$$
$$= C_2 + C_1(\beta+\gamma) + \beta^2 + \gamma^2 + \beta\gamma - \lambda(C_1 + \alpha + \beta + \gamma),$$

since the orders $\lambda - \lambda'' = \gamma - \alpha$.

And our result may be written in a symmetrical form if we write A for the common value of $\lambda + \alpha$, $\lambda' + \beta$, $\lambda'' + \gamma$, when it becomes

$$C_2 + C_1(\alpha+\beta+\gamma) + \alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta - A(C_1 + \alpha + \beta + \gamma),$$

or $$C_2 + C_1H_1 + H_2 - A(C_1 + H_1).$$

295. And, generally, if there be any number of relation columns, I have been led by a similar process to the following result: Let the terms in the relation columns be $\lambda, \lambda', \lambda''$, &c., μ, μ', μ'', &c., ν, ν', ν'', &c.; then we must have

$$\lambda + \alpha = \lambda' + \beta', \text{ &c., } \mu + \alpha = \mu' + \beta', \text{ &c. } \nu + \alpha = \nu' + \beta', \text{ &c.}$$

Let A, B, C denote the common values of these sums, and let H_1', H_2' denote the sum and sum of products as in Art. 272 of the quantities A, B, C; then the order of the system is

$$C_2 + C_1H_1 + H_2 - H_1'(C_1 + H_1) + H_2'.$$

This result may be stated as follows, in a way which leads us at once to foresee the answer to some other questions that

may be proposed as to the order of systems of these equations. In the case we are considering, the entire number of columns, counting the relation columns, is one more than the number of rows; *and the order of the system is that given by the rule of* Art. 272, *if we give a negative sign to the orders in the relation columns.* In like manner, when the number of columns, counting the relation columns, is equal to the number of rows, the system, by Prof. Cayley's theorem, represents a determinant whose order is that which we should obtain by calculating the order of the entire system considered as a determinant, the orders in the relation columns being taken negatively. And so no doubt if the entire number of columns exceeded the number of rows by two, the order of the system would be found by the same modification from the rule of Art. 285.

296. Let us now apply the rule just arrived at to the problem proposed in Art. 293. We consider the three ternary quantics of the order l, m, n respectively; and we regard these as depending upon two arbitrary parameters, the orders in these parameters being as follows; the coefficients of x^l, x^m, x^n, the highest powers of x, are of the orders λ, μ, ν; those of $x^{l-1}y$, $x^{l-1}z$ are of the orders $\lambda+\alpha$, $\lambda+\alpha'$, and so on, the orders of the coefficients increasing by α for every power of y, and by α' for every power of z. Then the terms in the first column consist first of $\frac{1}{2}(m+n-1)(m+n-2)$ terms whose orders are λ; $\lambda-\alpha$, $\lambda-\alpha'$; $\lambda-2\alpha$, $\lambda-\alpha-\alpha'$, $\lambda-2\alpha'$, &c.; secondly, of $\frac{1}{2}(n+l-1)(n+l-2)$ terms whose orders are μ; $\mu-\alpha$, $\mu-\alpha'$; &c., and thirdly of $\frac{1}{2}(l+m-1)(l+m-2)$ similar terms in ν. These may be taken for the numbers α, β, γ, &c of Art. 272. The numbers a, b, c, &c. of that article are 0, α, α' : 2α, $\alpha+\alpha'$, $2\alpha'$, &c., there being in all $\frac{1}{2}(l+m+n-1)(l+m+n-2)$ such terms. Lastly, the numbers A, B, C, &c. of the last article are found to consist of $\frac{1}{2}(l-1)(l-2)$ terms, $\mu+\nu$, $\mu+\nu-\alpha$, $\mu+\nu-\alpha'$; together with $\frac{1}{2}(m-1)(m-2)$ and $\frac{1}{2}(n-1)(n-2)$ corresponding terms in $\nu+\lambda$ and $\lambda+\mu$. In calculating I have found it convenient to throw the formula of the last article into the shape

$$\tfrac{1}{2}\{(C_1+H_1-H_1')^2-(S_2+S_2'-s_2)\},$$

where s_2 denotes the sum of the squares of the terms a, b, c, &c.

Also if $\phi(l) = Al^4 + Bl^3 + Cl^2 + Dl + E$, it is convenient to take notice that

$$\phi(l+m+n) + \phi(l) + \phi(m) + \phi(n) - \phi(l+m) - \phi(m+n) - \phi(n+l) = 12Almn(l+m+n) + 6Blmn + E.$$

I have thus arrived at the result, that the order of the system or number of the sets of values of the parameters is

$$\begin{aligned}
&\tfrac{1}{2}mn(mn-1)\lambda^2 + \tfrac{1}{2}nl(nl-1)\mu^2 + \tfrac{1}{2}lm(lm-1)\nu^2 \\
&+ \{(nl-1)(lm-1) - \tfrac{1}{2}(l-1)(l-2)\}\mu\nu \\
&+ \{(lm-1)(mn-1) - \tfrac{1}{2}(m-1)(m-2)\}\nu\lambda \\
&+ \{(nl-1)(nm-1) - \tfrac{1}{2}(n-1)(n-2)\}\lambda\mu \\
&+ mn\lambda\{lmn - l + 1 - \tfrac{1}{2}(m+n)\}(\alpha+\alpha') \\
&+ nl\mu\{lmn - m + 1 - \tfrac{1}{2}(n+l)\}(\alpha+\alpha') \\
&+ lm\nu\{lmn - n + 1 - \tfrac{1}{2}(l+m)\}(\alpha+\alpha') \\
&+ \tfrac{1}{2}lmn(lmn - l - m - n + 2)(\alpha^2 + \alpha'^2) + \tfrac{1}{2}lmn(2lmn - l - m - n + 1)\alpha\alpha'.
\end{aligned}$$

If the order of all the terms in the first equation be λ, in the second μ, in the third ν, we have only to make α and $\alpha' = 0$ in the preceding formula. In this case, supposing $\lambda = \mu = \nu = 1$, the order becomes

$$\begin{aligned}
&\tfrac{1}{2}(mn + nl + lm)(mn + nl + lm - 5) \\
&- \tfrac{1}{2}(l-1)(l-2) - \tfrac{1}{2}(m-1)(m-2) - \tfrac{1}{2}(n-1)(n-2),
\end{aligned}$$

and in particular if $l = m = n$, the order is

$$\tfrac{3}{2}n(n-1)(n^2+n-1).$$

This last result shows that if $U, U', U'', V, V', V'', W, W', W''$ be given homogeneous functions of (x, y, z) each of the order n, then the number of curve-triplets

$$U + \theta U' + \phi U'' = 0,\quad V + \theta V' + \phi V'' = 0,\quad W + \theta W' + \phi W'' = 0,$$

having 2 common points, is

$$= \tfrac{1}{2}n(n-1)(n^2+n-1).$$

297. Mr. S. Roberts applies to the problems of this Lesson a method directly applicable to binary quantics, since they can always be resolved into factors, and which extends to the case of ternary and higher quantics, for the question whether or not they can be so resolved does not affect the

problems here discussed, and the orders determined in the case of quantics which are the products of factors must be generally true. Thus, to determine the order of the resultant of two binary quantics of the degrees m, n; if the order of the terms in the first be λ, $\lambda+\alpha$, $\lambda+2\alpha$, &c., it may be resolved into the product of m factors $ax+by$, the orders of a and b being $\frac{\lambda}{m}$, $\frac{\lambda}{m}+\alpha$ respectively; similarly, for the second quantic; and the resultant is the product of mn factors, the order of each being $\frac{\lambda}{m}+\frac{\lambda'}{n}+\alpha$; and, therefore, mn times this number will be the order of the resultant. Now Mr. Roberts argues that we may deal in the same manner with the problem in Art. 277; that knowing, by Art. 272, the order of the matrix $\left\| \begin{matrix} a, & b \\ a', & b' \\ a'', & b'' \end{matrix} \right\|$ to be

$$\alpha^2+(\lambda+\mu+\nu)\alpha+\lambda\mu+\mu\nu+\nu\lambda,$$

the orders of the rows being supposed to be λ, $\lambda+\alpha$; μ, $\mu+\alpha$; ν, $\nu+\alpha$; then we may conclude that the order of the system of conditions for the simultaneous existence of three equations of orders l, m, n is

$$lmn\left\{\alpha^2+\alpha\left(\frac{\lambda}{l}+\frac{\mu}{m}+\frac{\nu}{n}\right)+\frac{\lambda\mu}{lm}+\frac{\mu\nu}{mn}+\frac{\nu\lambda}{nl}\right\}.$$

And in like manner, that the order of conditions for the co-existence of a system of $k+1$ binary equations is the product of their degrees multiplied by

$$\alpha^k+P_1\alpha^{k-1}+P_2\alpha^{k-2}+\ldots P_k,$$

where P_1, P_2, &c. are the sum, sum of products in pairs, &c. of the numbers $\frac{\lambda}{l}$, $\frac{\mu}{m}$, &c. And so more generally, the order of the conditions for the co-existence of any number of equations in any number of variables is derived from the order determined by Art. 272 for the co-existence of a system of linear equations. It is thus found that the order of conditions for the co-existence of $k+s-1$ homogeneous equations in s variables, in which the order of the coefficients of x^l, $x^{l-1}y$, $x^{l-1}z$, &c. is

λ, $\lambda+\alpha$, $\lambda+\beta$, &c. is the product of their degrees multiplied by

$$H_k + H_{k-1}P + H_{k-2}Q + \&c.,$$

where H_k has the same meaning as in Art. 272, and P, Q, &c. are the sum, sum of products in pairs, &c. of the numbers $\frac{\lambda}{l}$, $\frac{\mu}{m}$, &c. Thus, for instance, this formula applied to the case of ternary quantics gives the order of the conditions that a curve should have a cusp. We determine by the formula the order for the co-existence of U_1, U_2, U_3, $U_{11}U_{22} - U_{12}^2$, which system belongs either to cusps or double points on the line z, and we subtract the order for the co-existence of U_1, U_2, U_3, z, which belongs to the latter. The result is

$$12(n-1)(n-2)\lambda^2 + 8n(n-1)(n-2)(\alpha+\beta)\lambda + 2n(n-1)(n-2)(n+1)\alpha\beta + 2n(n-1)^2(n-2)(\alpha^2+\beta^2).$$

The problem of finding the order of conditions that two binary equations should have two common roots is discussed as follows: Consider first the simpler system, formed by taking two factors from each equation,

$$(ax+by)(a'x+b'y)(a''x+b''y)(a'''x+b'''y),$$

and we have the pair of conditions

$$(ab'')(a'b'') = 0, \quad (ab''')(a'b''') = 0,$$

whose order combined is $4(\lambda+\mu+\alpha)^2$; but from this we must subtract the irrelevant systems $(ab'')(ab''')$, $(a'b'')(a'b''')$, which reduces the order to $2(\lambda+\mu+\alpha)^2$. But if we take two factors from the first equation and one from the second, the system $(ab'')=0$, $(a'b'')=0$ is satisfied by $a''=0$, $b''=0$, whose order is $\mu(\mu+\alpha)$. Now since the number of ways in which two factors of the first equation may be combined with two of the second is $\frac{1}{2}l(l-1)\times\frac{1}{2}m(m-1)$, and the number of ways in which one of the second may be combined with two of the first is $\frac{1}{2}l(l-1)m$; the resulting order in general is

$$\tfrac{1}{2}lm(l-1)(m-1)\left(\frac{\lambda}{l}+\frac{\mu}{m}+\alpha\right)^2 + \tfrac{1}{2}lm(m-1)\frac{\lambda}{l}\left(\frac{\lambda}{l}+\alpha\right) + \tfrac{1}{2}ml(l-1)\frac{\mu}{m}\left(\frac{\mu}{m}+\alpha\right),$$

as in Art. 275.

By the same process of reasoning Mr. Roberts arrives at the order of the conditions (Art. 296) that three ternary quantics should have two points common, in the form

$$\{\tfrac{1}{2}lmn(l-1)(m-1)(n-1)+\Sigma\tfrac{1}{2}lmn(m-1)(n-1)\}\left\{\frac{\lambda}{l}+\frac{\mu}{m}+\frac{\nu}{n}+\alpha+\alpha'\right\}^2$$

$$+\Sigma\tfrac{1}{2}lmn\,(l-1)\left\{\frac{\lambda^2}{l^2}+\frac{\mu^2}{m^2}+\frac{\lambda\mu}{lm}+(\alpha+\alpha')\left(\frac{\lambda}{l}+\frac{\mu}{m}\right)+\alpha\alpha'\right\}.$$

In this way the order of conditions that a curve should have two double points is found to be

$$\tfrac{1}{2}(n-1)(n-2)^2(n+1)\{3\lambda+n(\alpha+\alpha')\}^2-\tfrac{1}{2}(n-1)(n-2)$$
$$\times\{15\lambda^2+10n(\alpha+\alpha')\lambda+n(n+6)\alpha\alpha'+2n(2n-3)(\alpha^2+\alpha'^2)\}.$$

Mr. Roberts investigates other problems by the same method; as, for instance, the order of conditions that four curves may have two points common, or that a surface may have a biplanar double point. For these I must refer to his paper.* I only give the following result: The order of conditions that three binary quantics should have two roots common is

$$\tfrac{1}{2}lmn(l-1)(m-1)(n-1)\left\{\frac{\lambda\mu}{lm}+\frac{\mu\nu}{mn}+\frac{\nu\lambda}{nl}+\alpha\left(\frac{\lambda}{l}+\frac{\mu}{m}+\frac{\nu}{n}\right)+\alpha^2\right\}^2$$

$$+\Sigma\tfrac{1}{2}lmn\,(m-1)(n-1)\left\{\frac{\lambda}{l}\left(\frac{\lambda}{l}+\alpha\right)\left(\frac{\mu}{m}+\frac{\nu}{n}+\alpha\right)^2\right\}$$

$$+\Sigma\tfrac{1}{2}lmn\,(n-1)\frac{\lambda\mu}{lm}\left(\frac{\lambda}{l}+\alpha\right)\left(\frac{\mu}{m}+\alpha\right).$$

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* Prof. Cayley, in the *Cambridge and Dublin Mathematical Journal*, vol. IV., p. 134, determined the order of a matrix with k rows and $k+1$ columns, in the particular case where each constituent is of the first degree. My own investigations were published, *Quarterly Journal*, vol. I., p. 246, and in the Appendix to my *Geometry of Three Dimensions*, second edition. After this Lesson was printed in the second edition Mr. Samuel Roberts communicated to me some extensions of the theory there developed, and his results have since been published, *Proceedings of the London Mathematical Society*, March, 1875.

LESSON XX.

APPLICATIONS OF SYMBOLICAL METHODS.

298. In this Lesson, which is supplementary to Lesson XIV., we wish to show how the symbolical notation there explained affords a calculus by means of which invariants and covariants can be transformed, and the identity of different expressions ascertained. In order to facilitate the reader's study of recent memoirs, we employ the notation explained, Art. 162, which is now almost exclusively used; to save the necessity of reference, we repeat what has been already said, and, in order to fix the ideas, we suppose the variables to be three, though the method is perfectly general. The variables then are x_1, x_2, x_3; if there are different sets of cogredient variables, such as the coordinates of different points, they are written y_1, y_2, y_3; z_1, z_2, z_3, &c. We use the abbreviation a_x for $a_1x_1 + a_2x_2 + a_3x_3$, a_y for $a_1y_1 + a_2y_2 + a_3y_3$; if we are only dealing with one set of variables so that no confusion is likely to arise, we sometimes suppress the suffix, and write a instead of a_x. The quantic of the n^{th} degree is symbolically written a_x^n, or $(a_1x_1 + a_2x_2 + a_3x_3)^n$; that is to say, a_1, a_2, a_3 are *umbral* symbols not regarded as having any meaning separately;* but a_1^n denotes the coefficient of x_1^n in the quantic, $a_1^{n-1}a_2$ that of $nx_1^{n-1}x_2$, and so on. And so generally any homogeneous function of the n^{th} degree in the letters a_1, a_2, a_3 may be replaced by a multiple sum of the coefficients of the quantic; any other function of these letters is not regarded as having a separate meaning. Other quantics may be denoted by b^p, c^q, &c., the symbols b_1, b_2, b, &c. being used in the same way. In the cases with which we principally deal the quantics are supposed to

* It has, however, been stated, Art. 163, that we can at any moment interpret a formula by substituting for a_1, a_2, a_3 differential symbols with regard to x_1, x_2, x_3.

be identical; and a^n, b^n, c^n, &c. are only different expressions for the same quantic.

We use (abc), (abd), &c., to denote determinants formed with the constituents a_1, a_2, a_3; b_1, b_2, b_3, &c. In order to express invariants or covariants of the quantic we take any number of such determinants and multiply them together; then evidently the product can be translated as a function of the coefficients of the given quantic, provided that the a symbols, b symbols, &c. respectively each occur n times. If not, we join in the product such powers of a_x, b_x, &c.—that is to say, of $(a_1x_1 + a_2x_2 + a_3x_3)$, &c.—as will make up the total number of a's, b's, &c. to n. We are then able to replace the symbolical letters by coefficients of the quantic, and the resulting product is a function of the coefficients and the variables, the latter entering in a degree equal to the sum of the orders of a_x, b_x, &c. in the symbolical product. It is easy to show that we obtain an invariant in the one case and a covariant in the other; and we refer to Clebsch's *Theorie der binären algebraischen Formen* for a formal proof that all invariants and covariants can be so expressed.* All this has been stated already (Art. 162). When a covariant is expressed in the manner explained, it is evident that its order in the coefficients is equal to the number of symbols a_1, b_1, &c., which enter into the determinant factors, and that its order in the variables is equal to the number of non-determinant factors a_x, b_x, &c.

Since the differential coefficients of $u = a_x^n$ are respectively $na_x^{n-1}a_1$, $na_x^{n-1}a_2$, $na_x^{n-1}a_3$, the equation of the polar, which is $\frac{1}{n}\left(y_1\frac{du}{dx_1} + y_2\frac{du}{dx_2} + y_3\frac{du}{dx_3}\right) = 0$, becomes $a_x^{n-1}a_y = 0$. Similarly, the second polar is $a_x^{n-2}a_y^2$, and so on.

* The principle of the proof is briefly this: we have seen, Art. 218, that from any invariant or covariant P of a single quantic, we can, by the operation $a'\frac{d}{da} + \&c.$, obtain a corresponding form Π for a system of two quantics, and that we can fall back from Π on P by making $a' = a$, &c. By repeated application of this principle, if the form P be of the r^{th} order in the coefficients, it may be considered as derived from a form Π belonging to r different functions, each of which may be symbolically written $(a_1x_1 + a_2x_2)^n$, $(b_1x_1 + b_2x_2)^n$, &c. Every form therefore of the r^{th} order for the single quantic has a corresponding form for r linear factors, and it is proved without difficulty that for the latter case the only invariant or covariant forms are those expressed as in the text.

299. Confining ourselves now to the case of two variables, a_x or a here stands for $a_1x_1 + a_2x_2$, (ab) stands for $a_1b_2 - a_2b_1$; and any covariant is expressed symbolically by a product $(ab)^\alpha (ac)^\beta (bd)^\gamma$ &c. multiplied by $a^p b^q c^r$ &c., the number of a's b's, &c., in the entire product being each n or a multiple of n. If p, q, r, &c. all vanish, the symbol denotes an invariant. Any symbol which simply changes sign by an interchange of a and b (as, for example, $(ab)^\alpha a^{n-\alpha} b^{n-\alpha}$, where α is odd) denotes an expression which vanishes identically (see Art. 153).

If we eliminate x_1, x_2 from the equations

$$a = a_1x_1 + a_2x_2, \quad b = b_1x_1 + b_2x_2, \quad c = c_1x_1 + c_2x_2,$$

we have

$$(A) \qquad a(bc) + b(ca) + c(ab) = 0,$$

an identity of the greatest use in transforming these expressions. Thus, for example, transposing $a(bc)$ to the other side and squaring, we have

$$(B) \qquad 2bc(ab)(ac) = b^2(ac)^2 + c^2(ab)^2 - a^2(bc)^2.$$

To illustrate the use of this, multiply by $a^{n-2}b^{n-2}c^{n-2}$, in order that each term may denote a covariant of an n-ic, and we have

$$2a^{n-2}b^{n-1}c^{n-1}(ab)(ac) = b^n a^{n-2}c^{n-2}(ac)^2 + c^n a^{n-2}b^{n-2}(ab)^2 - a^n b^{n-2}c^{n-2}(bc)^2.$$

Now, since a^n, b^n, c^n all equally denote the quantic, the three terms on the right-hand side of the equation are only different expressions for the same covariant; and we learn that the covariant $a^{n-2}b^{n-1}c^{n-1}(ab)(ac)$ is half the product of a^n (which is the quantic itself) by $b^{n-2}c^{n-2}(bc)^2$, which denotes the Hessian.

We can always (as has been stated Art. 163) interpret these symbolical expressions by supposing a_1, a_2, &c., to denote $\frac{d}{dx_1}$, $\frac{d}{dx_2}$, &c., by supposing that we operate on the product of several distinct quantics uvw, and by making the variables identical after differentiation. In this way a or $x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2}$, applied to any homogeneous function, only affects it with a numerical factor. If we thus interpret equation (B), $(ab)(ac)$

is Q, where Q is

$$\frac{d^2u}{dx_1^2}\left(\frac{du}{dx_2}\right)^2 - 2\frac{d^2u}{dx_1dx_2}\left(\frac{du}{dx_1}\right)\left(\frac{du}{dx_2}\right) + \frac{d^2u}{dx_2^2}\left(\frac{du}{dx_1}\right)^2;$$

and $(ab)^2$ is $2H$, where H is $\dfrac{d^2u}{dx_1^2}\dfrac{d^2u}{dx_2^2} - \left(\dfrac{d^2u}{dx_1dx_2}\right)^2$.

On the right-hand side of equation B, a^2, b^2, c^2 respectively operate on functions which have not been before differentiated, and therefore affect them with the numerical factor $n(n-1)$; on the left-hand side b, c operate on a function which has been once differentiated, and each affects it with the numerical factor $(n-1)$. Equation B therefore gives us $(n-1)\,Q = nUH$.

300. From equation B other useful formulæ may be derived. Squaring it, we have

$$(C)\quad a^4(bc)^4 + b^4(ca)^4 + c^4(ab)^4 = 2\{b^2c^2(ab)^2(ac)^2 + c^2a^2(bc)^2(ba)^2 + a^2b^2(ca)^2(cb)^2\}.$$

If this be applied to a quartic form, the three terms on the left-hand side are all different expressions for the same thing. So are also the three terms on the right; and we learn that the covariant $b^2c^2(ab)^2(ac)^2$ differs only by a numerical factor from the product of the quantic itself by the invariant $(bc)^4$.

Considering the four expressions

$$a = a_1x_1 + a_2x_2,\quad b = b_1x_1 + b_2x_2,\quad c = c_1x_1 + c_2x_2,\quad d = d_1x_1 + d_2x_2,$$

we at once verify the identity

$$(D)\qquad (ad)(bc) + (bd)(ca) + (cd)(ab) = 0,$$

which is also of very great use in the theory. We deduce from it

$$(E)\quad 2(bd)(cd)(ab)(ac) = (bd)^2(ac)^2 + (cd)^2(ab)^2 - (ad)^2(bc)^2.$$

$$(F)\quad (ad)^4(bc)^4 + (bd)^4(ca)^4 + (cd)^4(ab)^4 = 2\{(bd)^2(cd)^2(ab)^2(ac)^2 + (cd)^2(ad)^2(bc)^2(ba)^2 + (ad)^2(bd)^2(ca)^2(cb)^2\}.$$

To these may be added an identity which is really a different form of (D): this is

$$(G)\qquad a_xb_y - b_xa_y = (ab)(xy),$$

where (xy) denotes $(x_1y_2 - x_2y_1)$; and from it we deduce

$$(H)\qquad a_x^2a_y^2b_x^2b_y^2 = \tfrac{1}{2}\{a_x^2b_y^2 + a_y^2b_x^2 - (ab)^2(xy)^2\}.$$

301. A symbolical expression may be always so transformed that the highest power of any factor (ab) shall be even. For the signification of the symbol is not altered if we interchange the letters a and b; therefore

$$(ab)^{2m+1}\phi_1 = -(ab)^{2m+1}\phi_2 = \tfrac{1}{2}(ab)^{2m+1}(\phi_1 - \phi_2),$$

and by the help of equation A, $\phi_1 - \phi_2$ can always be so transformed as to be divisible by (ab). Thus $bc^{2m-1}(ab)^{2m-1}(ac)$ is at once reduced to $c^{2m}(ab)^{2m}$. For

$$bc^{2m-1}(ab)^{2m-1}(ac) = \tfrac{1}{2}c^{2m-1}(ab)^{2m-1}\{b(ac) - a(bc)\} = \tfrac{1}{2}c^{2m}(ab)^{2m}.$$

302. If we arrange symbolical products according to the number of determinant factors which they contain, we can, by these formulæ of reduction, reduce them to certain standard forms. If there is but a single factor (ab), the covariant vanishes identically (Art. 299), since it changes sign by an interchange of a and b. The possible forms with two factors are $(ab)^2$, $(ab)(ac)$, $(ab)(cd)$, of which the last vanishes identically, and, in Art. 299, we have expressed $(ab)(ac)$ in terms of $(ab)^2$. There is therefore only a single distinct covariant symbolically expressed with two determinant factors, viz. the Hessian $H = a^{n-2}b^{n-2}(ab)^2$. Any such form must denote either the Hessian or the product of the Hessian by a power of the quantic. So again, for three factors, the possible forms are $(ab)^3$, $(ab)^2(ac)$, $(ab)(ac)(ad)$, $(ab)(ac)(bc)$, $(ab)(ac)(bd)$; of these the first, fourth, and fifth vanish identically, and the second and third are found to be related as follows: Multiply equation B by $a^{n-3}b^{n-2}c^{n-2}d^{n-1}(ad)$; two of the terms on the right-hand side become identical, the third vanishes, and we have

$$2a^{n-3}b^{n-1}c^{n-1}d^{n-1}(ab)(ac)(ad) = 2a^{n-3}b^{n-2}c^{n}d^{n-1}(ab)^2(ad),$$

showing that the covariant expressed by the left-hand side is the product of the quantic itself by the covariant whose value is given (Art. 156). Generally every symbol having a pair of factors with a common letter $(ab)(ac)$ may be reduced to a more compact form by substituting for this pair their value from equation B, and so expressing the symbol by others in which this pair of factors is replaced by a single square factor. Symbols with four factors can be reduced to either of the forms

$(ab)^4$ or $(ab)^2(cd)^2$, which is H^2. For five factors the fundamental forms are $(ab)^4(ac)$ and $(ab)^2(ac)(de)^2$, the latter being the product of two distinct covariants. For six factors the forms are $(ab)^6$ (Art. 153), $(ab)^2(bc)^2(ca)^2$ (Art. 155), and $(ab)^2(cd)^2(ef)^2$, the last being the cube of the Hessian. We give a few examples of the reduction of these forms.

Ex. 1. To reduce $a^{n-4}b^{n-1}c^{n-1}d^{n-1}e^{n-1}(ab)(ac)(ad)(ae)$. Multiply together

$$2bc\,(ab)\,(ac) = b^2(ac)^2 + c^2(ab)^2 - a^2(bc)^2,$$
$$2de\,(ad)\,(ae) = d^2(ae)^2 + e^2(ad)^2 - a^2(de)^2;$$

multiply the product by $a^{n-4}b^{n-2}c^{n-2}d^{n-2}e^{n-2}$, and assemble the identical terms, then we have

$$4a^{n-4}b^{n-1}c^{n-1}d^{n-1}e^{n-1}\,(ab)\,(ac)\,(ad)\,(ae)$$
$$= 4c^ne^na^{n-4}b^{n-3}d^{n-2}\,(ab)^2\,(ad)^2 - 3a^nb^{n-2}c^{n-2}d^{n-2}e^{n-2}\,(bc)^2\,(de)^2$$

The last term is $-3UH^2$. The other term on the right-hand side is reduced by equation C. Multiply by $a^{n-4}b^{n-4}d^{n-4}$ the equation

$$a^4(bd)^4 + b^4(ad)^4 + d^4(ab)^4 = 2\{b^2d^2(ab)^2(ad)^2 + d^2a^2(bd)^2(ba)^2 + a^2b^2(ad)^2(bd)^2\},$$

then we have $2a^{n-4}b^{n-2}d^{n-2}(ab)^2(ad)^2 = d^na^{n-4}b^{n-4}(ab)^4$. The right-hand side therefore of the preceding equation reduces to $2SU^3 - 3UH^2$, where S is the covariant $a^{n-4}b^{n-4}(ab)^4$.

Ex. 2. To reduce $(ab)^4(ac)^2$. Multiply equation C by $a^{n-6}b^{n-6}c^{n-4}(ab)^2$, then we have, assembling identical terms,

$$c^na^{n-6}b^{n-6}(ab)^6 = 2a^{n-6}b^{n-4}c^{n-2}\,(ab)^4\,(ac)^2 + 2a^{n-4}b^{n-4}c^{n-4}\,(ab)^2\,(bc)^2\,(ca)^2.$$

Thus, if we call the standard forms $a^{n-6}b^{n-6}(ab)^6$ and $a^{n-4}b^{n-4}c^{n-4}(ab)^2(bc)^2(ac)^2$, M and T, we have $2a^{n-6}b^{n-4}c^{n-2}(ab)^4(ac)^2 = UM - 2T$.

Ex. 3. To reduce $(ab)^2(ac)^2(ad)^2$. Multiply together the three equations

$$2bc\,(ab)\,(ac) = c^2(ab)^2 + b^2(ac)^2 - a^2(bc)^2,$$
$$2cd\,(ac)\,(ad) = d^2(ac)^2 + c^2(ad)^2 - a^2(cd)^2,$$
$$2bd\,(ab)\,(ad) = b^2(ad)^2 + d^2(ab)^2 - a^2(bd)^2,$$

and multiply by $a^{n-6}b^{n-4}c^{n-4}d^{n-4}$, then we have

$$6a^{n-6}b^{n-2}c^{n-2}d^{n-2}\,(ab)^2\,(ac)^2\,(ad)^2$$
$$= -4d^na^{n-4}b^{n-4}c^{n-4}(ab)^2(bc)^2(ca)^2 + 6a^{n-6}b^{n-4}c^{n-2}d^n(ab)^4(ac)^2 - 3a^{n-4}b^{n-4}c^{n-2}d^{n-2}(ab)^4\,(cd)^2$$
$$= -4UT + 3U\,(UM - 2T) - 3SH = 3U^2M - 10UT - 3SH.$$

Ex. 4. To reduce $(ab)^2(ac)^2(cd)^2$. Multiply equation (C) by $(cd)^2$, and by $a^{n-4}b^{n-4}c^{n-6}d^{n-2}$, then we have

$$4a^{n-4}b^{n-2}c^{n-4}d^{n-2}\,(ab)^2\,(ac)^2\,(cd)^2$$
$$= 2a^nb^{n-4}c^{n-6}d^{n-2}\,(bc)^4\,(cd)^2 + a^{n-4}b^{n-4}c^{n-2}d^{n-2}\,(ab)^4\,(cd)^2 - 2a^{n-2}b^{n-2}c^{n-6}d^{n-2}\,(ac)^2(bc)^2(dc)^2$$

Multiply by three, and observe that in Ex. 3 we find that

$$6a^nb^{n-4}c^{n-6}d^{n-2}\,(bc)^4\,(cd)^2 - 6a^{n-2}b^{n-2}c^{n-6}d^{n-2}\,(ac)^2\,(bc)^2\,(dc)^2 = 4UT + 3SH,$$

and we have $\quad 6a^{n-4}b^{n-2}c^{n-4}d^{n-2}\,(ab)^2\,(ac)^2\,(cd)^2 = 2UT + 3SH.$

Ex. 5. To reduce $(ab)^2(cd)^2(ac)(bd)$. Multiply the equations

$$2ad\,(cd)\,(ca) = a^2(cd)^2 + d^2(ac)^2 - c^2(ad)^2,$$
$$2bc\,(cd)\,(bd) = b^2(cd)^2 + c^2(bd)^2 - d^2(bc)^2,$$

and multiply also by $a^{n-4}b^{n-4}c^{n-4}d^{n-4}(ab)^2$, and we have

$$-4a^{n-3}b^{n-3}c^{n-3}d^{n-3}(ab)^2(cd)^2(ac)(bd)$$
$$= a^{n-2}b^{n-2}c^{n-4}d^{n-4}(ab)^2(cd)^4 + 2a^{n-4}b^{n-2}c^{n-4}d^{n-2}(ab)^2(cd)^2(ac)^2$$
$$- 2a^{n-4}b^{n-4}c^{n-4}d^{n}(ab)^2(bc)^2(ca)^2$$
$$= SH - 2TU + 2a^{n-4}b^{n-2}c^{n-4}d^{n-2}(ab)^2(cd)^2(ac)^2;$$

therefore, by Ex. 4,

$$-12a^{n-3}b^{n-3}c^{n-3}d^{n-3}(ab)^2(cd)^2(ac)(bd) = 6SH - 4TU.$$

We stated (Art. 156) that the expression for the discriminant of a cubic was $(ab)^2(cd)^2(ac)(bd)$. The present example shows how the corresponding covariant of a quartic can be expressed in terms of the fundamental forms S, T, U, H.

303. If ϕ, ψ be covariants of the orders p, q in the variables, we may write these symbolically $\phi = (\phi_1 x_1 + \phi_2 x_2)^p$, $\psi = (\psi_1 x_1 + \psi_2 x_2)^q$, and we can obtain from them the series of covariants $\phi_x^{p-k}\psi_x^{q-k}(\phi\psi)^k$, where k has any value from 1 up to the least of the two numbers p and q. This operation, in German called *Ueberschiebung*, we shall call transvection, and the covariants generated we shall call transvectants of the two given covariants.

In the notation used Lesson XIV, $\overline{12}^k\phi\psi$ denotes a covariant differing only by a numerical factor from the transvectant; that is to say, if we denote differentiation with regard to x_1, x_2 by subindices, the series is

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \phi_{11}\psi_{22} - 2\phi_{12}\psi_{12} + \phi_{22}\psi_{11}, \quad \&c.$$

Thus the first in the series is the Jacobian, and if n be the lower of the orders of ϕ and ψ, the last in the series $\overline{12}^n\phi\psi$, is the result obtained by introducing differential symbols into the one quantic and operating on the other (Art. 139). We obtain transvectants of a single quantic by supposing ϕ and ψ to denote the same quantic. The transvectants then of odd order vanish, and those of even order form the series of covariants considered (Art. 141). The method of formation of covariants explained in this article has a prominent place in the proof given by Gordan and Clebsch, that the number of covariants of any form is always finite. Thus the first step in the proof is to show, as we shall presently do, that any covariant symbol formed with k letters a, b, c, &c. may be reduced to a transvectant of the original form combined with a covariant whose symbol contains only $k-1$ letters.

304. There is no difficulty in forming the transvectant, or any other derivative of a form ϕ symbolically expressed. The transvectants of ϕ, ψ, we have seen, are given by the formula $(\phi\psi)^k = (\xi_1\eta_2 - \xi_2\eta_1)^k$, where ξ_1, ξ_2 denote differentiation of ϕ, and η_1, η_2 differentiation of ψ. Now let the symbolic expression of ϕ be $\phi = Ma^r b^s$, &c., where M denotes the aggregate of the determinant factors. Then, since x_1, x_2 only enter in a, b, &c., we have

$$\xi_1 = a_1 \frac{d}{da} + b_1 \frac{d}{db} + \&c., \quad \xi_2 = a_2 \frac{d}{da} + b_2 \frac{d}{db} + \&c.$$

Similarly if $\psi = Nc^r d^s$ &c., we have

$$\eta_1 = c_1 \frac{d}{dc} + d_1 \frac{d}{dd} + \&c., \quad \eta_2 = c_2 \frac{d}{dc} + d_2 \frac{d}{dd} + \&c.$$

If we put these values into $(\xi_1\eta_2 - \xi_2\eta_1)^k$ we see that the symbolic expression for the transvectant consists of a group of terms, each of which contains all the determinant factors M, N of the two given forms, together with a function of the k^{th} order in the determinants (ac), (ad), (bc), (bd), &c. In particular the transvectants of ϕ combined with the original form u (or g^n, where g is a symbol not occurring in ϕ), are found by operating on ϕ with $\left\{(ag)\frac{d}{da} + (bg)\frac{d}{db} + (cg)\frac{d}{dc} + \&c.\right\}^k$.

Ex. To form the symbol for the Hessian of the Hessian. Here we have $\phi = a^{n-2}b^{n-2}(ab)^2$, $\psi = c^{n-2}d^{n-2}(cd)^2$, and we are to operate on $\phi\psi$ with $(\xi_1\eta_2 - \xi_2\eta_1)^2$; or with

$$\left\{(ac)\frac{d}{da}\frac{d}{dc} + (ad)\frac{d}{da}\frac{d}{dd} + (bc)\frac{d}{db}\frac{d}{dc} + (bd)\frac{d}{db}\frac{d}{dd}\right\}^2.$$

or, collecting terms, with

$$4\,(ac)^2\frac{d^2}{da^2}\frac{d^2}{dc^2} + 8\,(ac)(ad)\frac{d^2}{da^2}\frac{d}{dc}\frac{d}{dd} + 4\,(ac)(bd)\frac{d}{da}\frac{d}{db}\frac{d}{dc}\frac{d}{dd}.$$

It would be therefore

$$4\,(n-2)^2(n-3)^2(ab)^2(cd)^2(ac)^2 a^{n-4}b^{n-2}c^{n-4}d^{n-2}$$
$$+ 8\,(n-2)^3(n-3)(ab)^2(cd)^2(ac)(ad)\,a^{n-4}b^{n-2}c^{n-3}d^{n-3}$$
$$+ 4\,(n-2)^4(ab)^2(cd)^2(ac)(bd)\,a^{n-3}b^{n-3}c^{n-3}d^{n-3}.$$

It was shown, Art. 302, Ex. 4 and Ex. 5, that the first and third of these terms were expressible in the form $\alpha SH + \beta TU$, and the same thing is easily seen to be true of the second, if we substitute for $2cd\,(ac)\,(ad)$ its value $c^2(ad)^2 + d^2(ac)^2 - a^2(cd)^2$ from equation B. Thus we prove that the Hessian of the Hessian is expressible in the form $\alpha SH + \beta TU$, as was otherwise proved (Art. 211).

305. It is convenient to transform the formula given in the last article for $(u\phi)$, the transvectant of any form ϕ combined with the original. By equation A

$$(ag)\frac{d}{da}+(bg)\frac{d}{db}+(cg)\frac{d}{dc}+\&c.$$

$$=\frac{1}{a}(ag)\left\{a\frac{d}{da}+b\frac{d}{db}+\&c.\right\}-\frac{g}{a}\left\{(ab)\frac{d}{db}+(ac)\frac{d}{dc}+\&c.\right\},$$

where it will be observed that the operator $a\frac{d}{da}+b\frac{d}{db}+\&c.$ leaves the subject unchanged except by a numerical factor. Thus then if $\phi=Ma^pb^qc^r$ &c., where $p+q+r+\&c.=l$

$$(u\phi)=lM(ag)\,a^{p-1}b^qc^r\ldots g^{n-1}-g^nMa^{p-1}\left\{(ab)\frac{d}{db}+\&c.\right\}b^qc^r\ \&c.$$

Thus we have the expression divided into two parts, one in which we have the old determinant factors M together with the new factor (ag); the other in which the letter g does not enter into the determinant factors. Conversely, if we have a symbol in $k+1$ letters, one of which g only occurs once, this is immediately resolved into the transvectant of ϕ, u, together with a form whose symbol only contains k letters.

So, in like manner, the second transvectant with any form ϕ is

$$(u\phi)^2=\left\{(ag)\frac{d}{da}+(bg)\frac{d}{db}+\&c.\right\}^2\phi,$$

and therefore consists of a group of terms, each having all the determinant factors of ϕ, together with two new factors, each containing g. But, as before, this might be written

$$\frac{1}{a^2}\left[(ag)\left\{a\frac{d}{da}+b\frac{d}{db}+\&c.\right\}-g\left((ab)\frac{d}{db}+\&c.\right\}\right]^2\phi,$$

and therefore if $\phi=Ma^pb^q$ &c., we have

$$(u\phi)^2=l(l-1)\,M(ag)^2a^{p-2}b^q\ldots g^{n-2}+\psi,$$

where ψ consists of a group of terms into the determinant factors of which g enters only once; and therefore by the former part of this article, ψ can be reduced to a function whose

symbol does not contain g, together with the transvectant of such a function. Thus then any symbol of the form $\phi\,(ag)^2$ can be reduced to a function whose symbol does not contain g, together with the first or second transvectants of such functions. The same thing would be true of a function of the form $\phi\,(ag)\,(bg)$, as appears by writing the second transvectant in the form

$$\frac{1}{ab}\left[(ag)\left\{a\frac{d}{da}+\ldots\right\}-g\left\{(ab)\frac{d}{db}+(ac)\frac{d}{dc}+\ldots\right\}\right]$$

$$\times\left[(bg)\left\{a\frac{d}{da}+b\frac{d}{db}+\ldots\right\}-g\left\{(ba)\frac{d}{da}+(bc)\frac{d}{dc}+\ldots\right\}\right].$$

And so we see generally, that any symbol $\phi\,(ag)^\alpha\,(gb)^\beta$, where g occurs in all k times, can be reduced to the k^{th} transvectant of ϕ, together with terms in which g occurs only $k-1$ times, which again may be reduced in like manner. If, then, we arrange forms according to their order in the coefficients, it has been proved that the forms of any order consist either of forms of lower order multiplied by u or by powers of u, or else of transvectants obtained by combining u with forms of lower order.

306. We have just shewed that taking any one letter g in the symbol for a form, that form may be regarded as a transvectant of u combined with other covariants, and that the symbol for each of these other covariants will contain all the determinant factors of the original form, striking out those which contain g. So again taking any letter f in the symbol for any of these other covariants, we express that covariant as a transvectant of u combined with covariants whose symbol contains all the determinant factors of the original, striking out those which include the letters f and g. Proceeding thus we see that taking any of the letters, say, a, b, c, and considering the factors of the given form which contain these letters only, say $(ab)^\alpha\,(bc)^\beta\,(ca)^\gamma$, then the given form may be obtained by transvection from a form in these three letters having as a factor $(ab)^\alpha\,(bc)^\beta\,(ca)^\gamma$.

All that has been said in the last article applies equally if the original form, instead of being a covariant of a single

function, were a simultaneous covariant of several; that is to say, if instead of a^n, b^n, c^n all representing the same function U, they represent different functions U, V, W, &c. It remains true that if V be the function to which g refers, the form represents a transvectant of V and of other functions, whose symbols could be separated in like manner. It may be seen thus that if we had all possible covariants of two forms U, V separately considered, including in the series the powers and products of the simple covariants, the system of the simultaneous covariants of U, V is obtained by adding to these all possible transvectants of a form of one set with a form of the other. If the forms so obtained be combined by transvection with the covariants of a third fundamental form W, we obtain all possible forms of the system U, V, W, and so on. We refer to Clebsch, p. 186, or to Gordan, *Programm*, p. 18, for a proof that if the system of covariants of the separate forms U, V, W be finite, the number of distinct forms obtained by transvection in the manner described, and therefore the number of covariants of the system, is also finite.

307. We proceed now to give an outline of the method by which Gordan has shewed that the number of distinct forms for a quantic of the n^{th} degree is finite, and we shall shew that if this be true for a quantic of the degree $n-1$, it will be as true for one of the degree n. The symbol for a form belonging to a quantic of the $(n-1)^{\text{th}}$ degree being $Ma^pb^qc^r$ &c., where the a symbols, b symbols, &c. each occur $n-1$ times, it follows that if we multiply the symbol by abc &c. we shall have a form for a quantic of the n^{th} degree, since the a symbols, &c. will then each occur n times. We shall speak then of forms belonging to quantics of different degrees, as being the same, when the determinant factors in their symbols are the same, and when these symbols only differ by a power of abc &c. We propose to establish the following theorem, viz. that the forms for a quantic of the n^{th} degree consist either of the forms which had occurred already for a quantic of the degree $n-1$, or else of the mutual transvectants between such forms and the series of two-lettered forms $(ab)^2$, $(ab)^4$, &c. And in order

to establish this, we shall shew in the first place that every form for a quantic of the n^{th} degree either has in its symbol a factor $(ab)^p$ where p is not less than $\frac{1}{2}n$, or else is a form belonging also to a quantic of the $(n-1)^{\text{th}}$ degree; and with regard to the latter, we add the further restriction,* that if its symbol be $Ma^pb^qc^r$ &c., p the greatest of these indices shall not be less than $\frac{1}{2}n$. We shall prove this latter theorem by shewing that if it is true for forms of the m^{th} order in the coefficients, it is true for forms of the order $m+1$; and it evidently is true for forms of the first order, that is the quantic itself. Now it was shewn (Art. 305) that all forms of the order $m+1$ can be obtained from forms of the order m by transvection of such forms with u. Since this transvection only adds new determinant factors to the symbol without removing any of the old, it follows that every form of the order m having $(ab)^p$ in its symbol, will give rise by transvection to forms of the order $m+1$ having the same property. We need only consider therefore forms $Ma^pb^qc^r$ &c. Now we saw (Art. 304) that the k^{th} transvectant of such a form is got by operating k times with

$$\frac{1}{a}(ag)\left\{a\frac{d}{da}+b\frac{d}{db}+\&c.\right\}-\frac{g}{a}\left\{(ab)\frac{d}{db}+\&c.\right\},$$

that is to say, it will be of the form $M(ag)^k a^{p-k}b^qc^r\ldots g^{n-k}$, together with transvectants of the order $k-1$. The term which we have written will, in case k is as large as $\frac{1}{2}n$, contain the factor $(ag)^{\frac{n}{2}}$; and if k be less than $\frac{1}{2}n$, then since by hypothesis p is at least $\frac{1}{2}n$, $p-k$ the index of a, will be positive, and therefore the term will still be divisible by abc &c., and will therefore denote a form belonging to a quantic of lower or er, while $n-k$ the index of g, will exceed $\frac{1}{2}n$. We see then, that if forms of the order m in the coefficients consist only of the two classes we have named, the k^{th} transvectant of such form will consist only of the same two classes, provided this be true for the $(k-1)^{\text{th}}$ transvectant.

* This limitation is only necessary for the proof of the theorem, and is not used in any of its subsequent applications.

And so down step by step till we come to the 0 transvectant; that is to say, the product of the given form by a, for which it is obviously true. The theorem we have enunciated is therefore proved.

It follows from what has been just proved, that every invariant symbol must contain as a factor $(ab)^p$, where p is at least half n; for the other class of symbols, viz. those occurring in quantics of lower order, of necessity represent covariants, since they have the factors a, b, &c., each of which contains the variables.

Forms having $(ab)^p$ for a factor are (Art. 306) transvectants of forms having this as a factor, and it can be seen without difficulty that, except when $n = 4$, such forms will when p exceeds $\frac{1}{2}n$ be of lower order in the variables than the quantic itself. With this exception, then, it appears that forms for the n^{th} degree are simultaneous covariants of certain forms all of lower order than n; and therefore if the number be proved finite for numbers less than n, it is also finite for n (Art. 306). The case $n = 4$ requires a little speciality of treatment, for which we refer to Clebsch, p. 267.

308. It would evidently be convenient if a general symbolical expression could be given for the result of elimination between two equations. When one equation is simple it is easily seen that the eliminant between it and an equation of the n^{th} degree is $(\alpha a)\,(\alpha b)\,(\alpha c)$, &c., where the symbol α relates to the equation of the n^{th} degree, and the remaining symbols to the simple equation. I gave in 1853 a general formula for the resultant of a quadratic and an equation of the n^{th} degree (*Cambridge and Dublin Math. Jour.*,* IX., 32). The theorem was rediscovered by Clebsch in 1860, and extended by him to the

* In the same paper I investigated a formula for the discriminant of a binary quantic, and in this way obtained that for a quartic. Clebsch subsequently gave in his paper (*Crelle*, LIX.), "Ueber symbolische Darstellung algebraischer Formen," a rule for obtaining a general symbolic formula for the resultant of two binary quantics or for the discriminant of a binary quantic. The method of proceeding is to apply Cayley's form of Bezout's method of elimination, Art. 87, to two quantics written symbolically, but the resulting rule is, as might be expected, very complicated.

case of a system of any number of equations, one of the second, one of the n^{th}, and the rest of the first degree (*Crelle*, vol. LVIII). We give here Clebsch's investigation of the general theorem. To fix the ideas, I write only a system of four homogeneous equations in four variables, but it will be understood that the method is equally applicable to any number of variables. Let the equations then be

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0, \quad \beta = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 = 0,$$

$$U = u_{11} x_1^2 + 2u_{12} x_1 x_2 + \&c. = 0,$$

and $\phi = 0$ an equation of the n^{th} order in x_1, x_2, x_3, x_4 which may be written symbolically $(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4)^n = 0$. The method of elimination employed is to solve between the linear equations and the quadric, and substituting in ϕ the two systems of values found, to multiply the results together. Now we may in an infinity of ways combine the quadric with the linear equations multiplied by arbitrary factors, so as to obtain a result resolvable into factors: that is to say, so that

$$U + (\lambda_1 x_1 + \lambda_2 x_2 + \&c.)\ (\alpha_1 x_1 + \&c.) + (\mu_1 x_1 + \&c.)\ (\beta_1 x_1 + \&c.)$$
$$= (p_1 x_1 + \&c.)\ (q_1 x_1 + \&c.).$$

We shall imagine this transformation effected, but it will not be necessary to determine the actual values of λ_1, μ_1, &c. for it will be found that these quantities disappear from the result. Taking, then, the coefficient of any term $x_i x_k$ in the quadric, the equation written implies that we always have

$$2u_{ik} + (\alpha_i \lambda_k + \alpha_k \lambda_i) + (\beta_i \mu_k + \beta_k \mu_i) = p_i q_k + p_k q_i \cdots (A).$$

Instead then of solving between the quadric and the linear equations, we get the two systems of values by combining with the linear equations successively $p_1 x_1 + \&c. = 0$, $q_1 x_1 + \&c. = 0$. And by the theory of linear equations the resulting values of x_1, x_2, &c. are the determinants of the systems

$$\left\| \begin{matrix} p_1, & p_2, & p_3, & p_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \end{matrix} \right\|, \qquad \left\| \begin{matrix} q_1, & q_2, & q_3, & q_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \end{matrix} \right\|.$$

If then we substitute the first set of values in $a_1x_1 + \&c.$, we get the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ p_1, & p_2, & p_3, & p_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \end{vmatrix},$$

which we may write $(a_1p_2\alpha_3\beta_4)$. The result of elimination may thus be written symbolically $R = (a_1p_2\alpha_3\beta_4)^n (b_1q_2\alpha_3\beta_4)^n$. We use in the second factor the symbol b instead of a, for the reason explained (Art. 162), in order to obtain powers of the coefficients of ϕ; but it is understood that the b symbols have exactly the same meaning as the a, since after expansion we equally replace the products $a^ia^ka^la^m$, $b^ib^kb^lb^m$, by the corresponding coefficient of ϕ, a_{iklm}. We may then write the result of elimination in the more symmetrical form

$$2R = (a_1p_2\alpha_3\beta_4)^n (b_1q_2\alpha_3\beta_4)^n + (a_1q_2\alpha_3\beta_4)^n (b_1p_2\alpha_3\beta_4)^n,$$

for this after expansion will be only double the former expression.

Let us now write

$$(a_1p_2\alpha_3\beta_4)\ (a_1q_2\alpha_3\beta_4) = A,$$

$$(b_1p_2\alpha_3\beta_4)\ (b_1q_2\alpha_3\beta_4) = B,$$

$$(a_1p_2\alpha_3\beta_4)\ (b_1q_2\alpha_3\beta_4) + (a_1q_2\alpha_3\beta_4)\ (b_1p_2\alpha_3\beta_4) = 2C,$$

then R may be easily expressed in terms of A, B, C. For we have

$$2R = \{C + \sqrt{(C^2 - AB)}\}^n + \{C - \sqrt{(C^2 - AB)}\}^n,$$

or
$$R = C^n + \frac{n(n-1)}{1.2} C^{n-2} (C^2 - AB)$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} C^{n-4} (C^2 - AB)^2 + \&c.$$

309. We proceed now to examine more closely the expressions for A, B, C, and to get rid of the quantities p and q which we have introduced, so that the result may be expressed in terms of the coefficients of the given quadric.

Now A, which is the product of two determinants, may be written as a single determinant,

$$\begin{vmatrix}
p_1q_1 & \frac{1}{2}(p_1q_2+p_2q_1) & \frac{1}{2}(p_1q_3+p_3q_1) & \frac{1}{2}(p_1q_4+p_4q_1) & \alpha_1 & \beta_1 & a_1 \\
\frac{1}{2}(p_1q_2+p_2q_1) & p_2q_2 & \frac{1}{2}(p_2q_3+p_3q_2) & \frac{1}{2}(p_2q_4+p_4q_2) & \alpha_2 & \beta_2 & a_2 \\
\frac{1}{2}(p_1q_3+p_3q_1) & \frac{1}{2}(p_2q_3+p_3q_2) & p_3q_3 & \frac{1}{2}(p_3q_4+p_4q_3) & \alpha_3 & \beta_3 & a_3 \\
\frac{1}{2}(p_1q_4+p_4q_1) & \frac{1}{2}(p_2q_4+p_4q_2) & \frac{1}{2}(p_3q_4+p_4q_3) & p_4q_4 & \alpha_4 & \beta_4 & a_4 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots\ldots & & \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \ldots\ldots & & \\
a_1 & a_2 & a_3 & a_4 & \ldots\ldots & &
\end{vmatrix}$$

multiplied however by $(-1)^{m-1}$, where m is the number of variables, that is to say, in the present case, 4.

For, every element of this determinant must contain a constituent from each of the last three rows and columns; it is therefore of the first degree in the terms p_1q_1, $\frac{1}{2}(p_1q_2+p_2q_1)$, &c.; and if the coefficient of any of these terms be examined, it will be found, according to the number of variables, to be either the same, or the same with sign changed, as in the product of the two determinants. Now in this determinant we are to substitute from equation (A),

$$p_1q_1 = u_{11} + \alpha_1\lambda_1 + \beta_1\mu_1,$$

$$\tfrac{1}{2}(p_1q_2+p_2q_1) = u_{12} + \tfrac{1}{2}(\alpha_1\lambda_2 + \alpha_2\lambda_1) + \tfrac{1}{2}(\beta_1\mu_2 + \beta_2\mu_1), \text{ &c.}$$

But when this change has been made, if we subtract from each of the first of the four rows and columns the α row and column each multiplied by $\frac{1}{2}\lambda_1$, and the β row and column each multiplied by $\frac{1}{2}\mu_1$, &c., the additional terms disappear, and the determinant reduces to

$$\begin{vmatrix}
u_{11} & u_{12} & u_{13} & u_{14} & \alpha_1 & \beta_1 & a_1 \\
u_{21} & u_{22} & u_{23} & u_{24} & \alpha_2 & \beta_2 & a_2 \\
u_{31} & u_{32} & u_{33} & u_{34} & \alpha_3 & \beta_3 & a_3 \\
u_{41} & u_{42} & u_{43} & u_{44} & \alpha_4 & \beta_4 & a_4 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots\ldots & & \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \ldots\ldots & & \\
a_1 & a_2 & a_3 & a_4 & \ldots\ldots & &
\end{vmatrix}.$$

Clebsch denotes the above determinant, in which the matrix of the discriminant of a quadric is bordered by rows and

columns, α, β, &c., by the abbreviation (see p. 18)

$$\begin{pmatrix} \alpha, & \beta, & a \\ \alpha, & \beta, & a \end{pmatrix},$$

the upper line denoting the columns, and the lower the rows by which the matrix is bordered. Thus, then, we find for any number of variables

$$A = (-1)^{m-1} \begin{pmatrix} \alpha, & \beta, & \ldots a \\ \alpha, & \beta, & \ldots a \end{pmatrix}.$$

In like manner $$B = (-1)^{m-1} \begin{pmatrix} \alpha, & \beta, & \ldots b \\ \alpha, & \beta, & \ldots b \end{pmatrix}.$$

And in the same way

$$C = (-1)^{m-1} \begin{pmatrix} \alpha, & \beta, & \ldots a \\ \alpha, & \beta, & \ldots b \end{pmatrix}.$$

Now it was proved, Ex. 2, p. 32, that $\begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} b \\ b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}^2 = \Delta \begin{pmatrix} a, & b \\ a, & b \end{pmatrix}$, where Δ is the discriminant of the quadric. And it is proved in the same way, in general, that

$$\begin{pmatrix} \alpha, & \beta, & \ldots a \\ \alpha, & \beta, & \ldots a \end{pmatrix}\begin{pmatrix} \alpha, & \beta, & \ldots b \\ \alpha, & \beta, & \ldots b \end{pmatrix} - \begin{pmatrix} \alpha, & \beta, & \ldots a \\ \alpha, & \beta, & \ldots b \end{pmatrix}^2 = \Delta \begin{pmatrix} \alpha, & \beta, & \ldots a, & b \\ \alpha, & \beta, & \ldots a, & b \end{pmatrix}.$$

If, then, we call the last written function D, we have

$$C^2 - AB = -\Delta D,$$

and the formula of last Article becomes

$$R = C^n - \frac{n(n-1)}{1.2} C^{n-2} D\Delta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} C^{n-4} D^2 \Delta^2 - \&c.$$

310. As an example of the application of this formula we give Clebsch's investigation of the equation of the system of inflexional tangents to a cubic (*Crelle*, 59), remarking in the first place that formulæ of reduction corresponding to those given (Arts. 299, 300), exist for quantics in any number of variables. Thus for ternary quantics the most useful are

$$a(bcd) - b(acd) + c(abd) - d(abc) = 0,$$

$$(abc)(ade) + (abd)(aec) + (abe)(acd) = 0;$$

to which may be added the corresponding equations for contravariant symbols (Art. 160)

$$P(abc) = a(abc) + b(\alpha ca) + c(\alpha ab),$$

$$(\alpha ab)(\alpha cd) + (abc)(\alpha ad) + (\alpha ca)(abd) = 0,$$

where P is $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$. To come now to the problem proposed, the equation of the system of inflexional tangents to a cubic is got (*Higher Plane Curves*, Art. 74) by eliminating between the equation of the curve and those of its first and second polars, and one of these equations being linear and another quadratic, the formula of the last article is applicable. By Art. 308 the eliminant is $4C^3 - 3ABC$, where A, B, C are respectively $\begin{pmatrix} u, a \\ u, a \end{pmatrix}$, $\begin{pmatrix} u, b \\ u, b \end{pmatrix}$, $\begin{pmatrix} u, a \\ u, b \end{pmatrix}$. But we have proved, p. 18, that

$$\begin{pmatrix} u, a \\ u, a \end{pmatrix} = -\tfrac{3}{2} u \begin{pmatrix} a \\ a \end{pmatrix} - \tfrac{1}{4} a^2 H, \quad \begin{pmatrix} u, a \\ u, b \end{pmatrix} = -\tfrac{3}{2} u \begin{pmatrix} a \\ b \end{pmatrix} - \tfrac{1}{4} abH.$$

The result therefore is when cleared of fractions

$$4\left\{abH + 6u\begin{pmatrix} a \\ b \end{pmatrix}\right\}^3 - 3\left\{abH + 6u\begin{pmatrix} a \\ b \end{pmatrix}\right\}\left\{a^2 H + 6u\begin{pmatrix} a \\ a \end{pmatrix}\right\}\left\{b^2 H + 6u\begin{pmatrix} b \\ b \end{pmatrix}\right\}.$$

But remembering that a^3 and b^3 are each u, this may be divided by u, and thus gives

$$uH^3 + 18PH^2 + 108QHu + 216Ru^2,$$

where $$P = 3a^2 b^2 \begin{pmatrix} a \\ b \end{pmatrix} - ua\begin{pmatrix} a \\ a \end{pmatrix} - ub\begin{pmatrix} b \\ b \end{pmatrix},$$

$$Q = 4ab\begin{pmatrix} a \\ b \end{pmatrix}^2 - ab\begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} b \\ b \end{pmatrix} - a^2\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} b \\ b \end{pmatrix} - b^2\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix},$$

$$R = 4\begin{pmatrix} a \\ b \end{pmatrix}^3 - 3\begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} b \\ b \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

In the above formula, as at p. 17, H denotes the determinant formed with the unreduced second differential coefficients; but if we suppose that, as in the *Higher Plane Curves*, each coefficient has been cleared of the numerical factor six, we must write in the above for H, $216H$; also, since P, Q, R involve the second differential coefficients in the 2nd, 4th, 6th degrees respectively, these will be 6^2, 6^4, 6^6 times the corresponding P, Q, R. Making these substitutions, the eliminant becomes

$$uH^3 + 3PH^2 + 3QHu + Ru^2.$$

To reduce this further observe that $a\begin{pmatrix} a \\ a \end{pmatrix} = -3H$; for $\begin{pmatrix} a \\ a \end{pmatrix}$ expanded is $-(u_{22}u_{33} - u_{23}^2)\, a_1^2 +$ &c., but $a_1^2 a$ (Art. 298) is u_{11} &c.,

on making which substitutions, the truth of what has been stated appears. In a similar manner it is easily seen that $a^2\binom{a}{b} = -Hb$, from which at once follows that $a^2b^2\binom{a}{b} = -Hu$. We have then $P = 3Hu$.

In order to calculate Q, it appears from what we have just stated that

$$ab\binom{a}{a}\binom{b}{b} = a\binom{a}{a} \times b\binom{b}{b} = 9H^2, \text{ and } a^2\binom{a}{b}\binom{b}{b} = -Hb\binom{b}{b} = 3H^2.$$

Again we can show that $ab\binom{a}{b}^2 = 3H^2$ for (Art. 33, Ex. 2) $\binom{a}{b}^2 = \binom{a}{a}\binom{b}{b} - H\binom{a, b}{a, b}$, and by actual expansion it is easily seen that $ab\binom{a, b}{a, b} = 6H^2$. Collecting the terms we have $Q = -3H^2$. Thus as the two terms $3PH^2$, $3QHu$ cancel each other, the eliminant is seen to be divisible by u and reduces to $H^3 + Ru$.

311. It remains to calculate R. Now let us in the first place observe, that if it be required to differentiate H, which is $\begin{vmatrix} u_{11}, & u_{12}, & u_{13} \\ u_{21}, & u_{22}, & u_{23} \\ u_{31}, & u_{32}, & u_{33} \end{vmatrix}$ with regard to x_1, it follows from what was said at the end of Art. 34, that the result is

$$(u_{22}u_{33} - u^2_{32})a_{111} + \&c.,$$

that is to say, the result is $-a_1\binom{a}{a}$. Taking then the determinant $\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & a_1 \\ u_{21}, & u_{22}, & u_{23}, & a_2 \\ u_{31}, & u_{32}, & u_{33}, & a_3 \\ b_1, & b_2, & b_3, & \end{vmatrix}$, and multiplying the fourth column by $\binom{a}{a}$ and the fourth row by $\binom{b}{b}$, we have

$$\binom{a}{b}\binom{a}{a}\binom{b}{b} = \binom{H}{H};$$

that is to say, we mean by $\binom{H}{H}$ the result of substituting the differential coefficients of H for a_1, a_2, a_3 and for b_1, b_2, b_3 in

the determinant just written. We have also (Art. 33, Ex. 2)

$$\binom{a}{b}^3 - \binom{a}{a}\binom{b}{b}\binom{a}{b} = -H\binom{a}{b}\binom{a,b}{a,b}.$$

But if we take $a_1\binom{a}{a}$ and differentiate it by the same process as that already employed, we find that the second differential of H with regard to x_1, x_1, is $a_1b_1\binom{a,b}{a,b}$, and so on. Thus it will be seen that $\binom{a}{b}\binom{a,b}{a,b}$ is $h\binom{h}{h}$, by which we mean the result of substituting in $\binom{a}{a}$ for a_1, a_2, a_3 symbols of differentiation applicable only to H. We may get another expression for $\binom{a}{b}\binom{a,b}{a,b}$. It is the coefficient of λ in the expansion of the Hessian of $u+\lambda H$. For, that Hessian is found by substituting in the determinant which expresses the Hessian, for each second differential coefficient u_{11}, $u_{11}+\lambda a_1b_1\binom{a,b}{a,b}$, and it is easily seen that the coefficient of λ in this expansion is as stated. But if we remember (*Higher Plane Curves*, Art. 225) that that coefficient is $-2Su$, and that $\binom{H}{H}$ is what is called Θ (*Higher Plane Curves*, Art. 231), we identify the result now obtained with that there given (Art. 232, Ex. 1), viz.

$$H^3 - 5Su^2H + u\Theta.$$

312. In the theory of double tangents to plane curves explained (*Higher Plane Curves*, Art. 384), it is necessary to calculate the result of substituting in the successive emanants $\alpha_3u_2-\alpha_2u_3$, $\alpha_1u_3-\alpha_3u_1$, $\alpha_2u_1-\alpha_1u_2$ for x_1, x_2, x_3, and to show that each result is of the form $P_n+Q_n(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3)^2$. We give as a further illustration of the use of symbolical methods the application of them to the calculation of Q_2, &c. These results of substitution may be symbolically expressed $a^{n-2}(a\alpha u)^2$, $a^{n-3}(a\alpha u)^3$, &c., where a only is a symbol. These expressions may be reduced by the help of a general formula for $(a\alpha u)^2$, which is found as follows, as in *Higher Plane Curves*, Art. 390:

This square differs only in sign from the determinant

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & a_1, & \alpha_1, & u_1 \\ u_{21}, & u_{22}, & u_{23}, & a_2, & \alpha_2, & u_2 \\ u_{31}, & u_{32}, & u_{33}, & a_3, & \alpha_3, & u_3 \\ a_1, & a_2, & a_3, & . & . & . \\ \alpha_1, & \alpha_2, & \alpha_3, & . & . & . \\ u_1, & u_2, & u_3, & . & . & . \end{vmatrix}.$$

But this determinant is reduced as at p. 18, so that the outside row and column become $0, 0, 0, -a, -\alpha, -u$; and we have

$$(a\alpha u)^2 = a^2 \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} + \alpha^2 \begin{pmatrix} a \\ a \end{pmatrix} - 2a\alpha \begin{pmatrix} a \\ \alpha \end{pmatrix} - u \begin{pmatrix} a, \alpha \\ a, \alpha \end{pmatrix}.$$

In the calculations we have to make, we suppose that the values of x_1, x_2, x_3 make u vanish. To calculate then Q_2, multiply this equation by a^{n-2}, the first term on the right-hand side has u for a factor and vanishes, $a^{n-2}\begin{pmatrix} a \\ a \end{pmatrix}$ is $-3H$, and $a^{n-1}\begin{pmatrix} a \\ \alpha \end{pmatrix}$ is $-\alpha H$. Thus we have $Q_2 = -\alpha^2 H$.

Again $Q_3 = a^{n-1}\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}(a\alpha u) + \alpha^2 a^{n-3}\begin{pmatrix} a \\ a \end{pmatrix}(a\alpha u) - 2a^{n-2}\alpha\begin{pmatrix} a \\ \alpha \end{pmatrix}(a\alpha u)$,

but the first and last terms vanish identically; and the middle term reduces to $-\alpha^2(\alpha H u)$, for we have seen that the differentials of the Hessian are $-a^{n-3}a_1\begin{pmatrix} a \\ a \end{pmatrix}$, &c.

Again, for Q_4 we have to calculate

$$a^{n-4}\left\{a^2\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} + \alpha^2\begin{pmatrix} a \\ a \end{pmatrix} - 2a\alpha\begin{pmatrix} a \\ \alpha \end{pmatrix}\right\}^2 = a^n\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}^2 - 4a^{n-1}\alpha\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}\begin{pmatrix} a \\ \alpha \end{pmatrix}$$
$$+ 4a^{n-2}\alpha^2\begin{pmatrix} a \\ \alpha \end{pmatrix}^2 + 2a^{n-2}\alpha^2\begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} - 4a^{n-3}\alpha^3\begin{pmatrix} a \\ \alpha \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix} + a^{n-4}\alpha^4\begin{pmatrix} a \\ a \end{pmatrix}^2.$$

The first term vanishes, the second is $4\alpha^2\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}H$, the third is $-4\alpha^2\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}H$, as may be seen from last Article, and since $a^{n-2}\begin{pmatrix} a \\ a \end{pmatrix}$ is $-3H$, the fourth is $-6\alpha^2\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}H$. Thus we see that the result is divisible by α^2, giving

$$Q_4 = -6\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}H - 4\alpha\begin{pmatrix} H \\ \alpha \end{pmatrix} + a^{n-4}\alpha^2\begin{pmatrix} a \\ a \end{pmatrix}^2.$$

313. As final illustration of the use of symbolic methods we give typical expositions of binary quantics, in the first place, of even degree by means of three independent quadratic covariants (Clebsch, p. 414). Let the three independent quadratics be

$$k = k_x^2 = k_1^2 x_1^2 + 2k_1 k_2 x_1 x_2 + k_2^2 x_2^2, \quad l = l_x^2, \quad m = m_x^2,$$

and their Jacobians $\kappa = (lm)\, l_x m_x$, $\lambda = (mk)\, m_x k_x$, $\mu = (kl)\, k_x l_x$, and denote by accented letters symbols for the same form.

Then for $p = p_x^2$ any fourth quadric (compare p. 181) we have $(p\kappa)^2 = (lm)(mp)(lp)$, and writing $D = -(lm)(mk)(kl)$, the identity

$$\begin{vmatrix} k, & k_1^2, & k_1 k_2, & k_2^2 \\ l, & l_1^2, & l_1 l_2, & l_2^2 \\ m, & m_1^2, & m_1 m_2, & m_2^2 \\ p, & p_1^2, & p_1 p_2, & p_2^2 \end{vmatrix} = 0$$

becomes $p_x^2 D = k\,(p\kappa)^2 + l\,(p\lambda)^2 + m\,(p\mu)^2$.

Thus, if the required even quantic f be symbolically expressed by a_x^{2n}, we find f expressed by means of k, l, m by replacing p by a in this symbol and raising it to the power n:

$$\begin{aligned} fD^n = &\{(a\kappa')^2 k + (a\lambda')^2 l + (a\mu')^2 m\} \\ &\times \{(a\kappa'')^2 k + (a\lambda'')^2 l + (a\mu'')^2 m\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\times \{(a\kappa^{(n)})^2 k + (a\lambda^{(n)})^2 l + (a\mu^{(n)})^2 m\}. \end{aligned}$$

When this product is expanded the coefficients of its terms $k^\alpha l^\beta m^\gamma$ are evidently invariants, thus the expression is typical (p. 249). The quadric relation which subsists between k, l, m enables us to go back from this ternary to a binary form: introducing in so doing only the invariants of the three quadratics.

If we retain k, l, m as three variables the symbol α_y^n defined by $\alpha_y = (a\kappa)^2 y_1 + (a\lambda)^2 y_2 + (a\mu)^2 y_3$ is evidently that of a unique curve of the n^{th} order, which is determined by the $2n$ points of f, which are now given on a conic. The additional relations of this curve to the conic by which its uniqueness is secured arise from the fact that, if the conic be written symbolically $\beta_y^2 = \beta_y'^2$, then their covariant $(\alpha\beta\beta')^2\, \alpha_y^{n-2}$ vanishes identically*;

* F. Lindemann, *Bulletin de la Soc. Math. de France*, t. v. 1877. For $n > 3$ this specializes the curve, thus he notices that the conditions for $n = 4$ limit the ternary quartic to admit of being the sum of five fourth powers, cf. Note, p. 151.

or again, that there is an infinity of triangles conjugate to this conic which are inscribed in the polar conic of y relative to the curve $\alpha_y^n = 0$; in other words, that all the conical polars of this curve are apolar to the conic β_y^2.

314. The exposition, however, can be made to involve invariants of lower degrees by using as its basis instead of the quadratics their Jacobians. In fact, as in Art. 193, we can see that $R_{123}u_4 = D_{41}J_{23} + D_{42}J_{31} + D_{43}J_{12} = u_1R_{234} - u_2R_{341} + u_3R_{412}$, or, in the present notation, that

$$k(a\kappa)^2 + l(a\lambda)^2 + m(a\mu)^2 = (ak)^2\kappa + (al)^2\lambda + (am)^2\mu$$

whence
$$\begin{aligned} fD^n = &\{(ak')^2\kappa + (al')^2\lambda + (am')^2\mu\} \\ &\times \{(ak'')^2\kappa + (al'')^2\lambda + (am'')^2\mu\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\times \{(ak^{(n)})^2\kappa + (al^{(n)})^2\lambda + (am^{(n)})^2\mu\}. \end{aligned}$$

If n be even, we can multiply out and replace all powers of κ, λ, μ by functions of the same order in k, l, m without denominator. If n be odd, the product of all factors but one gives us even powers of κ, λ, μ which can be so treated, and the last factor when expressed by k, l, m introduces a new factor D upon f. Thus the coefficients are the n^{th} transvectants of f on $k^\alpha l^\beta m^\gamma$, where $\alpha + \beta + \gamma = \frac{1}{2}n$, and are accordingly of much lower order than the others, which were its transvectants of the same order on $\kappa^\alpha\lambda^\beta\mu^\gamma$ where $\alpha + \beta + \gamma = \frac{1}{2}n$.

It is in this manner that Clebsch developes the expression for the sextic by its quadric covariants when E does not vanish. Each covariant symbolically contained in

$$E^2f = a_x^2\{(al)^2\lambda + (am)^2\mu + (an)^2\nu\}\{(al')^2\lambda + (am')^2\mu + (an')^2\nu\}$$

is found as a linear function of l, m, n by means of the invariants A, I_2, I_3, D', and, on expanding, the products of the second order of λ, μ, ν are replaced by quadric functions of l, m, n without denominator.

315. In a similar manner Dr. F. Lindemann (*Math. Ann.* XXIII., p. 133) has given the typical exposition of forms of the order $3p$ by means of four independent cubic covariants in connection with the formulæ of Art. 219*b*, and the consequent geometrical reference to a twisted cubic without the canonical reduction employed in Arts. 219*c*, *d*.

In fact, now denoting the cubics by k_x^3, l_x^3, m_x^3, n_x^3, and the forms $-J_{234}$, J_{341}, $-J_{412}$, J_{123} by κ_x^3, λ_x^3, μ_x^3, ν_x^3, it is easily seen that for any other cubic a_x^3 we have $(a\kappa)^3=(mn)(nl)(lm)(al)(am)(an)$, and writing $(k\kappa)^3=(l\lambda)^3=$ &c. $=\Delta$, that

$$\begin{aligned}\Delta a_x^3 &= (a\kappa)^3 k_x^3 + (a\lambda)^3 l_x^3 + (a\mu)^3 m_x^3 + (a\nu)^3 n_x^3\\ &= (ak)^3 \kappa_x^3 + (al)^3 \lambda_x^3 + (am)^3 \mu_x^3 + (an)^3 \nu_x^3.\end{aligned}$$

Whence we find the $3p^{ic}$ f by raising this to the degree p

$$\begin{aligned}\Delta^p a_x^{3p} = &\{(a\kappa')^3 k + (a\lambda')^3 l + (a\mu')^3 m + (a\nu')^3 n\}\\ &\times \{(a\kappa'')^3 k + (a\lambda'')^3 l + (a\mu'')^3 m + (a\nu'')^3 n\}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\times \{(a\kappa^{(p)})^3 k + (a\lambda^{(p)})^3 l + (a\mu^{(p)})^3 m + (a\nu^{(p)})^3 n\},\end{aligned}$$

or

$$\begin{aligned}= &\{(ak')^3 \kappa + (al')^3 \lambda + (am')^3 \mu + (an')^3 \nu\}\\ &\times \{(ak'')^3 \kappa + (al'')^3 \lambda + (am'')^3 \mu + (an'')^3 \nu\}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\times \{(ak^{(p)})^3 \kappa + (al^{(p)})^3 \lambda + (am^{(p)})^3 \mu + (an^{(p)})^3 \nu\}.\end{aligned}$$

Either form expanded is a function of degree p in four variables with invariant coefficients. Taking the point $z_1=k_x^3$, $z_2=l_x^3$, &c. as variable with x its locus is a twisted cubic, and the coordinates of the osculating planes determined by x are $u_1=\kappa_x^3$, $u_2=\lambda_x^3$, &c. The equations of three quadrics through this curve or of three quadrics touched by all its osculating planes can be easily found from the obvious relation

$$\Delta\,(xx')^3 = k_x^3 \kappa_{x'}^3 + l_x^3 \lambda_{x'}^3 + m_x^3 \mu_{x'}^3 + n_x^3 \nu_{x'}^3,$$

and the others which follow from it as successive polars. When found these equations enable us to return to a binary system introducing only invariants of the cubics.

Thus the equation of the surface, whose intersections with the twisted cubic are the $3p$ points of the binary quantic, is either $\alpha_z^p=0$ where $\alpha_z=(a\kappa)^3 z_1+(a\lambda)^3 z_2+(a\mu)^3 z_3+(a\nu)^3 z_4$, or $\beta_u^p=0$ where $\beta_u=(ak)^3 u_1+(al)^3 u_2+(am)^3 u_3+(an)^3 u_4$, this tangential form involving invariants of lower order.

This surface $\alpha_z^p=0$ is unique, and if we denote by $u_\rho^2=0$, $u_\sigma^2=0$, $u_\tau^2=0$, the three quadrics touched by all osculating planes of the twisted cubic, it is shown by Dr. Lindemann that this is because the three equations $\alpha_\rho^2\alpha_z^{p-2}=0$, $\alpha_\sigma^2\alpha_z^{p-2}=0$, $\alpha_\tau^2\alpha_z^{p-2}=0$ are identically true, and therefore $\alpha_z^p=0$ is conjugate, or apolar, to all the quadrics which can be inscribed in the developable surface of the twisted cubic.

NOTES.

HISTORY OF DETERMINANTS. (Page 1).

THE following historical notices are taken from Baltzer's *Theory of Determinants*; and from the sketch prefixed to Spottiswoode's *Elementary Theorems relating to Determinants*. The first idea of determinants is due to Leibnitz, as Dirichlet has pointed out. In Leibnitz's letter to L'Hôpital, 28 April, 1693 (Leibnitz's *Mathematical Works*, published by Gerhardt, vol. II., p. 239), is to be found the first example of the formation of these functions, and of their application to the solution of linear equations; the double suffix notation (p. 7) is employed, and he expresses his conviction of the fertility of his idea. But nowhere else in his writings is there to be found any proof that he sought to draw any new fruits from his discovery; and the method was lost until re-discovered by Cramer in 1750. Cramer, in his *Introduction à l'Analyse des lignes Courbes* (Appendix), has exhibited the determinants arising from linear equations in the case of two and three variables, and has indicated the law according to which they would be formed in the case of a greater number. The rule of signs by the method of displacements (p. 6) is given by Cramer. The equivalence of the other method by permutations of suffixes was afterwards proved by Bezout and Laplace. In the *Histoire de l'Académie Royale des Sciences*, Année 1764 (published in 1767), Bezout has investigated the degree of the equation resulting from the elimination of unknown quantities from a given system of equations, and has at the same time noticed several cases of determinants, without however entering upon the general law of formation, or the properties of these functions. The *Histoire de l'Académie*, An. 1772, part II. (published in 1776), contains papers by Laplace and Vandermonde relating to determinants of the second, third, fourth, &c. orders. The former, in discussing a system of simultaneous differential equations, has given the law of formation, and shown that when two rows or columns are interchanged, the sign of the determinant is changed, and that when two are identical, the determinant vanishes. The latter employs a notation in substance identical with that which, after Mr. Sylvester, we have called the umbral notation, and explained p. 8. In his Memoir on Pyramids (*Mémoires de l'Académie de Berlin*, 1773), Lagrange made an extensive use of determinants of the third order, and demonstrated that the square of such a determinant can itself be expressed as a determinant. The next impulse to the study was given by Gauss, *Disquisitiones Arithmeticæ*, 1801, who showed, in the case of the second and third orders. that the product of two determinants is a determinant, and very completely discussed the case of determinants of the second order arising from quadratic functions, *i.e.* of the form $b^2 - ac$. In 1812 Binet published a memoir on this subject (*Journal de l'Ecole Polytechnique*, tome IX., cahier 16), in which he establishes the principal theorems for determinants of the second, third and fourth orders, and applies them to geometrical problems. The next volume of the same series contains a paper, written at the same time, by Cauchy, on functions which only change sign when the variables which they contain are transposed. The second part of this paper refers immediately to determinants, and contains a large number of very general theorems. Cauchy introduced the name "determinants," already applied by

Gauss to the functions considered by him, and called by him "determinants of quadratic forms." In 1826 Jacobi took possession of the new calculus, and the volumes of Crelle's *Journal* contain brilliant proofs of the power of the instrument in the hand of such a master. By his memoirs in 1841, *De formatione et proprietatibus determinantium* and *De determinantibus functionalibus* (*Crelle*, vol. XXII.), determinants first became easily accessible to all mathematicians. Of later papers on this subject, perhaps the most important are Cayley's papers on Skew Determinants (*Crelle*, vol. XXXII. and XXXVIII.). Of elementary treatises on this subject, I have to mention Spottiswoode's *Elemeniary Theorems relating to Determinants*, London (1851); Brioschi, *La teorica dei determinanti*, Pavia, 1854; and Baltzer, *Theorie und Anwendung der Determinantem*, Leipzig, 1857; fifth edition, 1881. French translations both of Brioschi's and Baltzer's works have been published.

COMMUTANTS. (Page 8).

In connection with the umbral notation may be explained what is meant by *commutants*, which are but an extension of the same idea. It is easy to see what, according to the rule of the umbral notation, is meant by $\xi, \eta, \quad \xi^2, \xi\eta, \eta^2, \quad \xi, \eta, \quad \xi^2, \xi\eta, \eta^2,$ if we write for brevity ξ, η, for $\frac{d.}{dx}, \frac{d}{dy}$. We compound the partial constituents in each column in order to find the factors in the product we want to form, and take the sum with proper signs of all possible products obtained by permuting the terms in the lower row. Thus the first example denotes $\xi^2.\eta^2 - \xi\eta.\xi\eta$, which is the Hessian: the second denotes $\xi^4.\xi^2\eta^2.\eta^4 - \xi^4.\xi\eta^3.\xi\eta^3$ &c., which is the ordinary cubinvariant of a quartic.

Again, since multiplication is performed by addition of indices, it will be readily understood that we can equally form commutants where the partial constituents are combined by addition instead of by multiplication. Thus, considering the quantics

$$(a_2, a_1, a_0\,\}\!\!\{\, x, y)^2, \quad (a_4, a_3, a_2, a_1, a_0\,\}\!\!\{\, x, y)^4,$$

the invariants in the last two examples may be written $1, 0, \quad 2, 1, 0,$ which $1, 0, \quad 2, 1, 0,$ expanded are $a_2a_0 - a_1a_1$; $a_4a_2a_0 - a_4a_1a_1 +$ &c.

All these commutants with only two rows may be written as determinants, but it is a natural extension of the above notation to form commutants with more than two rows, such as $\xi, \eta, \quad 1, 0, \quad \xi^2, \xi\eta, \eta^2$. These all denote the sum of $\xi, \eta, \quad 1, 0, \quad \xi^2, \xi\eta, \eta^2.$ $\xi, \eta, \quad 1, 0, \quad \xi^2, \xi\eta, \eta^2.$ $\xi, \eta, \quad 1, 0, \quad \xi^2, \xi\eta, \eta^2.$ a number of products, each product consisting of as many factors as there are columns in the commutant and each factor being formed by compounding the constituents of the same column; and where we permute in every possible way the constituents in each row after the first. Thus the first and second examples denote the same thing, namely, the quadrinvariant of a quartic expressed in either of the forms $\xi^4.\eta^4 - 4\xi^3\eta.\xi\eta^3 + 3\xi^2\eta^2.\xi^2\eta^2$ or $a_4a_0 - 4a_3a_1 + 3a_2a_2$, while the third example $\xi^3.\xi^4\eta^4.\eta^3 -$ &c. denotes the cubinvariant of an octavic given at length, Art. 155.

We have seen that the two invariants of a binary quartic can be expressed as commutants, but it will be found impossible to express in the same way the discriminant of a cubic. Thus the leading term in it being $a_3^2a_0^2$ or $\xi_2\xi_3\eta_3\eta_3$, we are naturally led to expect that it might be the commutant $\xi, \eta, \xi, \eta,$ but this commu- $\xi, \eta, \xi, \eta,$ $\xi, \eta, \xi, \eta,$

tant, instead of giving the discriminant, will be found to vanish identically. It may, however, be made to yield the discriminant by placing certain restrictions on the permutations which are allowable. For further details I refer to the papers of Messrs. Cayley and Sylvester in the *Cambridge and Dublin Mathematical Journal*, 1852.

ON RATIONAL FUNCTIONAL DETERMINANTS.* (Page 14).

The determinants considered Ex. 5, 6, are particular cases of the important form

$$\left| \begin{matrix} \phi(x), \psi(x) \dots \\ \phi(y), \psi(y) \dots \\ \dots\dots\dots\dots \end{matrix} \right|,$$

where $\phi(x)$, $\psi(x)$ denote rational integral functions of x, and $\phi(y)$, $\phi(z)$, &c., the same functions of y, z, &c. respectively. Such a determinant may be briefly denoted by its top line $| \phi(x), \psi(x) \dots |$. Thus the determinant Ex. 5 may be written $| 1, x, x^2 \dots x^{n-1} |$. This last determinant we have seen has for its value

$$(-1)^{\frac{1}{2}n(n-1)} \zeta^{\frac{1}{2}}(x, y, z \dots);$$

by which notation Prof. Sylvester denotes the continued product of the differences

$$(x-y)(x-z)(x-w) \dots \times (y-z)(y-w) \dots \times (z-w) \text{ &c.}$$

This alternate product is of the nature of a square root: its square we know is a symmetrical function of x, y, z, &c., and is unaltered by any permutation of these variables; but itself has two values corresponding to the different arrangement of the variables, its sign being altered if we permute any two of the variables. The function $| 1, x, \dots x^{n-1} |$ was suggested by Cauchy as a symbolic representation of a determinant, viz. expanding it as the sum of a series of terms $\pm y^1 z^2 w^3 \dots$, and changing the exponents into suffixes, or the term into $x_0 y_1 z_2 w_3 \dots$, we have the development of the determinant $\left| \begin{matrix} x_0, x_1 \dots x_{n-1} \\ y_0, y_1 \dots y_{n-1} \\ \dots\dots\dots \end{matrix} \right|$.

It may be further remarked in passing, that any rational function of the variables x, y, &c., which, however the variables are permuted, has only two values, must be of the form $P + Q\zeta^{\frac{1}{2}}$, where P, Q are symmetric functions of the variables.

Returning now to the general determinant $| \phi(x), \psi(x) \dots |$, it obviously contains $\zeta^{\frac{1}{2}}$ as a factor, for on the supposition $x = y$, it vanishes as having two rows the same, and is therefore divisible by $x-y$; and similarly with regard to every other difference. Let us then in particular examine $| x^\alpha, x^\beta, x^\gamma \dots |$, which we may call $| \alpha, \beta, \gamma, \dots |$ in order to find the value of the remaining factor. If α be the least of these exponents, we may divide each row by $x^\alpha, y^\alpha, \dots$ respectively, so that we can at once reduce the investigation to that of the case where $\alpha = 0$.

In the following we employ a method given by Jacobi, *De Functionibus Alternantibus, Crelle* 22, (1841); depending on the consideration of the determinant $\left| \frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c} \dots \right|$. For convenience we work with the case of three variables, but it will be seen that the process is perfectly general. Consider then the equation which is obviously true

$$\left| \begin{matrix} \frac{1}{x-a}, & \frac{1}{x-b}, & \frac{1}{x-c} \\ \frac{1}{y-a}, & \frac{1}{y-b}, & \frac{1}{y-c} \\ \frac{1}{z-a}, & \frac{1}{z-b}, & \frac{1}{z-c} \end{matrix} \right| = - \frac{\zeta^{\frac{1}{2}}(x, y, z)\, \zeta^{\frac{1}{2}}(a, b, c)}{(x-a)(x-b)(x-c)(y-a)\ (y-b)(y-c)(z-a)(z-b)(z-c)}$$

* This note is, in substance, Professor Cayley's.

and expand each side by descending powers of x, y, z. We have

$$\frac{1}{x-a} = \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \&c.;$$

whence, on the left-hand side, the term multiplying the reciprocal of $x^\alpha y^\beta z^\gamma$ is

$$\begin{vmatrix} a^{\alpha-1}, & b^{\alpha-1}, & c^{\alpha-1} \\ a^{\beta-1}, & b^{\beta-1}, & c^{\beta-1} \\ a^{\gamma-1}, & b^{\gamma-1}, & c^{\gamma-1} \end{vmatrix}.$$

In order to expand the right-hand side, observe first that if H_1, H_2 &c. have the same meaning as at Art. 272, then

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{1}{x^3} + \frac{H_1}{x^4} + \frac{H_2}{x^5} + \&c.,$$

as is easy to see by multiplying together the expansions for $\frac{1}{x-a}$ &c. We have also

$$\zeta^3(x, y, z) = -\begin{vmatrix} 1, & x, & x^2 \\ 1, & y, & y^2 \\ 1, & z, & z^2 \end{vmatrix}.$$

Hence, the right-hand side is $\zeta^3(a, b, c)$ multiplied by

$$\left| \frac{1}{x^3} + \frac{H_1}{x^4} + \&c., \quad \frac{1}{x^2} + \frac{H_1}{x^3} + \&c., \quad \frac{1}{x} + \frac{H_1}{x^2} + \&c. \right|,$$

and the term multiplying the reciprocal of $x^\alpha y^\beta z^\gamma$ is $\zeta^3(a, b, c)$ multiplied by

$$\begin{vmatrix} H_{\alpha-3}, & H_{\alpha-2}, & H_{\alpha-1} \\ H_{\beta-3}, & H_{\beta-2}, & H_{\beta-1} \\ H_{\gamma-3}, & H_{\gamma-2}, & H_{\gamma-1} \end{vmatrix}.$$

We have thus

$$\begin{vmatrix} a^{\alpha-1}, & b^{\alpha-1}, & c^{\alpha-1} \\ a^{\beta-1}, & b^{\beta-1}, & c^{\beta-1} \\ a^{\gamma-1}, & b^{\gamma-1}, & c^{\gamma-1} \end{vmatrix} = \zeta^3(a, b, c) \begin{vmatrix} H_{\alpha-3}, & H_{\alpha-2}, & H_{\alpha-1} \\ H_{\beta-3}, & H_{\beta-2}, & H_{\beta-1} \\ H_{\gamma-3}, & H_{\gamma-2}, & H_{\gamma-1} \end{vmatrix},$$

which we may write $(\alpha-1, \beta-1, \gamma-1) = \zeta^3(a, b, c)\, H(\alpha-3, \beta-3, \gamma-3)$.

We may verify this equation by writing $\alpha=1$, $\beta=2$, $\gamma=3$, observing that $\zeta^3(a, b, c) = -(0, 1, 2)$, that H_{-2}, H_{-1} vanish and that H_0 is 1.

If we write $\alpha=1$, and for β, γ write $\beta+1, \gamma+1$, we have

$$(0, \beta, \gamma) = \zeta^3 H(-2, \beta-2, \gamma-2) = \zeta^3 \begin{vmatrix} H_{-2}, & H_{-1}, & H_0 \\ H_{\beta-2}, & H_{\beta-1}, & H_\beta \\ H_{\gamma-2}, & H_{\gamma-1}, & H_\gamma \end{vmatrix}.$$

But since H_{-2}, H_{-1} vanish, and $H_0=1$, the last determinant reduces to $\begin{vmatrix} H_{\beta-2}, & H_{\beta-1} \\ H_{\gamma-2}, & H_{\gamma-1} \end{vmatrix}$.

Thus we have finally $(0, \beta, \gamma) = \zeta^3 (H_{\beta-2}H_{\gamma-1} - H_{\beta-1}H_{\gamma-2})$. As an example, taking $\beta=1$, we get $(0, 1, \gamma) = -H_{\gamma-2}\zeta^3$, a formula which includes that of Ex. 6, p. 15.

We may also consider determinants involving the square roots of rational functions

$$\begin{vmatrix} \sqrt{\{\phi(x)\}}, & x\sqrt{\{\psi(x)\}}, & \dots \\ \sqrt{\{\phi(y)\}}, & y\sqrt{\{\psi(y)\}}, & \dots \\ \vdots & \vdots & \end{vmatrix},$$

but these, although presenting themselves in the theories of Elliptic and Abelian functions, have been but little studied.

HESSIAN. (Page 17).

The name was given by Sylvester after Professor Otto Hesse, who made much use of the functions in question, which he called functional determinants. They are a particular case of those studied under the same name by Jacobi (*Crelle*

vol. XXII.), the constituents of which are the differentials of a series of n homogeneous functions in n variables. It is so convenient to have short distinctive names for the functions of which we have repeatedly occasion to speak, that I have followed Sylvester in calling the former Hessians, the latter Jacobians, see Art. 88.

SYMMETRIC FUNCTIONS. (Page 56).

The rules for the weight and order of symmetric functions are Prof. Cayley's. The formula, Art. 59, I have taken from Serret's *Lessons on Higher Algebra*. The differential equation, Art. 60, is an anticipation of the differential equation for invariants, of which I speak, Art. 143. Brioschi (see M. Roberts, *Quarterly Journal*, vol. IV. p. 168), remarked that the operation $\{y\delta_x\}$ (Art. 65), expressed in terms of the roots, is $\frac{d}{d\alpha} + \frac{d}{d\beta} + \&c.$

ELIMINATION. (Page 66).

The name 'eliminant' was introduced I think by Professor De Morgan; I believe I have done wrong in using a second appellation when a name to which there was no objection was already in use. The older name 'resultant' was employed by Bezout, *Histoire de l'Académie de Paris*, 1764. The method of elimination by symmetric functions is due to Euler (*Berlin Memoirs*, 1748). The reduction of the resultant to that of a linear system was made simultaneously by Euler (*Berlin Memoirs*, 1764) and Bezout (*Paris Memoirs*, 1764). The theorem as to the degree of the resultant is Bezout's. The method used in Art. 74 of forming symmetric functions of the common values of a system of two or more equations is Poisson's (see *Journal de l'Ecole Polytechnique*, Cahier XI.). Sylvester gave his mode of elimination in the *Philosophical Magazine* for 1840, and called it 'dialytical,' because the process as it were *dissolves* the relations which connect the different combinations of powers of the variables and treats them as simple independent quantities. Cayley's statement of Bezout's method is to be found, *Crelle*, vol. LIII., p. 366. Sylvester's results in Art. 91 are to be found in the *Cambridge and Dublin Mathematical Journal* for 1852, vol. VII., p. 68; and Cayley's general theory (Art. 92, &c.) in the same Journal, vol. III., p. 116. It was noticed by Lagrange, that when two equations have two sets of common roots, the differential of the resultant with respect to the last term vanishes (see *Berlin Memoirs*, 1770). Sylvester showed, in January, 1853, that the same was true of all the differentials, *Cambridge and Dublin Mathematical Journal*, vol. VIII., p. 64. He showed at the same time, that the common roots were given by the ratios of the differentials. The proof in Art. 99 is, I believe, my own. The theorem, Art. 99, is Jacobi's *Crelle*, vol. XV., p. 105. In this part I have made some use of the *Treatise on Elimination* by Faá de Bruno. The theorem of Art. 102 is Prof. Cayley's.

DISCRIMINANTS. (Page 98).

The word 'discriminant' was introduced by Sylvester in 1852, *Cambridge and Dublin Mathematical Journal*, vol. VI., p. 52. The word 'determinant' had been previously used, and had come to have a perplexing variety of significations. The theorem referred to, Note, Art. 111, was the basis of my investigations (*Cambridge and Dublin Mathematical Journal*, 1847 and 1849) on the nature of cones circumscribing surfaces having multiple lines. If the equation of a surface be $b_0 + b_1x + b_2x^2 + \&c.$, and if xy be a double line, y must be contained by b_0 in the second and b_1 in the first degree. The discriminant with respect to x is a tangent cone which has y^2 for a factor.

BEZOUTIANTS. (Page 107).

It has been shown (Art. 85) that the resultant of two equations of the n^{th} degree is expressed by Bezout's method as a symmetrical determinant. This may be considered (Art. 118) as the discriminant of a quadratic function which Sylvester has called the Bezoutiant of the system. When the quantics are the two differentials of the same quantic, then if we resolve the Bezoutiant into a sum of squares (Art. 165), the number of negative squares in this sum will indicate the number of pairs of imaginary roots in the quantic. The number of negative squares is found by adding (as in Art. 46) λ to each of the terms in the leading diagonal of the matrix of the Bezoutiant, and then determining by Des Cartes' rule the number of negative roots in the equation for λ. The result of this method is to substitute for the leading terms in Sturm's functions, terms which are symmetrical with respect to both ends of the quantic; that is to say, which do not alter when for x we substitute its reciprocal (see Sylvester's Memoir, *Philosophical Transactions*, 1853, p. 513).

LINEAR TRANSFORMATIONS. (Page 108).

The germ of the principle of invariance may be traced to Lagrange, who, in the *Berlin Memoirs*, 1773, p. 265, established the invariance of the discriminant of the quadratic form $ax^2+2bxy+cy^2$, when for x is substituted $x+\lambda y$. Gauss, in his *Disquisitiones Arithmeticæ* (1801), investigated very completely the theory of the general linear transformation as applied to binary and ternary quadratic forms, and, in particular, established the invariance of their discriminants. This property of invariance was shown to belong to discriminants generally by the late Professor Boole, who, in a remarkable paper, *Cambridge Mathematical Journal*, 1841, vol. III., pp. 1, 106, applied it to the theory of orthogonal substitutions. He there showed how to form simultaneous invariants of a system of two functions of the same degree by performing on the discriminant of one of them the operation $a'\frac{d}{da}+b'\frac{d}{db}+$ &c. Boole's paper led to Cayley's proposing to himself the problem to determine *à priori* what functions of the coefficients of an equation possess this property of invariance. He found that it was not peculiar to discriminants, and he discovered other functions of the coefficients of an equation at first called by him 'hyper-determinants,' possessing the same property. Cayley's first results were published in 1845 (*Cambridge Mathematical Journal*, vol. IV., p 193). From this discovery of Cayley's, the modern algebra which forms the subject of the bulk of this volume may be said to take its rise. Among the first invariants distinct from discriminants, which were thus brought to light, were the quadrinvariants of binary quantics, and in particular the invariant S of a quartic. Mr. Boole next discovered the other invariant T of a quartic, and the expression of the discriminant in terms of S and T (*Cambridge Mathematical Journal*, vol. IV., p. 208). It is worthy of notice that both the functions S and T had been used by Eisenstein (*Crelle*, 1844, XXVII, p. 81) in his expression for the general solution of a quartic, but their property of invariance was unknown to him, as well as the expression for the discriminant in terms of them. Cayley next (1846) published the symbolical method of finding invariants, explained in Lesson XIV. (*Cambridge and Dublin Mathematical Journal*, vol. I., p. 104, *Crelle*, vol. XXX.). The next important paper was by Aronhold, 1849 (*Crelle*, vol. XXXIX., p. 140), in which the existence of the invariants S and T of a ternary cubic was demonstrated. Early in 1851 Mr. Boole reproduced, with additions, his paper on Linear Transformations (*Cambridge and Dublin Mathematical Journal*, vol. VI., p. 87), and Sylvester began his series of papers in the same Journal on the Calculus of Forms, after which discoveries followed

in rapid succession. I can scarcely pretend to be able to assign to their proper authors the merits of the several steps; and, as between Messrs. Cayley and Sylvester, perhaps these gentlemen themselves, who were in constant communication with each other at the time, would now find it hard to say how much properly belongs to each. To Mr. Boole (*Cambridge and Dublin Mathematical Journal*, vol. VI., p. 95, January, 1851) is, I believe, due the principle that in a binary quantic the operative symbols $\frac{d}{dy}$, $-\frac{d}{dx}$ may be substituted for x and y. The principle was extended to quantics in general by Sylvester, to whom is to be ascribed the general statement of the theory of contravariants, *Cambridge and Dublin Mathematical Journal*, (1857), vol. VI., p. 291; although particular applications of contravariants had previously been made in Geometry in the theory of Polar Reciprocals, and in the theory of ternary quadratic forms by Gauss (*Disquisitiones Arithmeticæ*, Art. 267), who gives the reciprocal under the name of the adjunctive form, and establishes its invariance under what he calls the "transformed substitution." Sylvester also remarked that we might not only replace contravariant by operative symbols, but also by the actual differentials $\frac{du}{dx}$, $\frac{du}{dy}$, &c. To Boole I would ascribe the principle (Art. 126) that invariants of emanants are covariants of the quantic (1842), *Cambridge Mathematical Journal*, vol. III., p. 110, though Boole's methods were generalized by Sylvester, *Cambridge and Dublin Mathematical Journal*, vol. VI., p. 190. Some of the first steps in the general theory of covariants may thus be ascribed to Boole, though a remarkable use of such a function had been made by Hesse in determining the points of inflexion of plane curves. I had myself been led to study the same functions both for curves and surfaces, in ignorance of what Hesse had done (*Cambridge and Dublin Mathematical Journal*, vol. II., p. 74). The discovery of evectants (Art. 134) is Hermite's, *Cambridge and Dublin Mathematical Journal*, vol. VI., p. 292. In Cayley's first paper he gave a system of partial differential equations satisfied by invariants of functions linear in any number of sets of variables. The partial differential equations (Art. 149) satisfied by the invariants and covariants of binary quantics were, as far as I know, first given in print by Sylvester (*Cambridge and Dublin Mathematical Journal*, vol. VII., p. 211). Sylvester there acknowledges himself to have been indebted to an idea communicated to him in conversation by Cayley; and he also speaks of having heard it said that Aronhold also was in possession of a system of differential equations. These are not made use of in Aronhold's paper (*Crelle*, vol. XXXIX.) already referred to, but he refers, *Crelle*, vol. LXII., to a communication made by him in 1851 to the Philosophical Faculty at Königsberg, which, if it ever appeared in print, I have not seen. Very probably there may be other parts of the theory to which Aronhold may justly lay claim. After the publication in *Crelle*, vol. XXX., of Cayley's paper, in which the symbolical method of forming invariants was fully explained, Aronhold worked at the theory in Germany simultaneously with the labours of Cayley and Sylvester in England; and the mastery of the subject exhibited by his papers leads me to suppose that of some of the principles he must be able to claim independent if not prior discovery. The method in which the subject is introduced (Art. 121) is taken from his paper (*Crelle*, vol. LXII). I refer in a note on next page to the valuable paper by Hermite (*Cambridge and Dublin Mathematical Journal*, vol. IX., p. 172), in which the theorem of reciprocity was established, which had at first suggested itself to Sylvester, but was hastily rejected by him, and in which the whole theory of quintics received important additions. Mixed concomitants are Sylvester's (*Cambridge and Dublin Mathematical Journal*, vol. VII p. 80). The theorem, Art. 135, is Cayley's and Sylvester's. The application of symmetric functions to the invariants of binary quantics was, I believe, first made in the Appendix to my *Higher Plane Curves* (1852). The method (Art. 138) of thence finding

conditions for systems of equalities between the roots is Cayley's (*Philosophical Transactions*, 1857, p. 703). With regard to the subject generally, reference must be made to the important series of papers by Sylvester, beginning in the sixth volume of the *Cambridge and Dublin Mathematical Journal*; to a series of papers on Quantics published by Cayley in the *Philosophical Transactions*; and to Aronhold's Memoir on Invariants (*Crelle*, vol. LXII.). The name 'invariant,' as well as much of the rest of the nomenclature, is Sylvester's.

CANONICAL FORMS. (Page 150).

The name is Hermite's; the theory explained in this Lesson is Sylvester's, see a paper (*Philosophical Magazine*, November, 1851) published separately, with a supplement, in the same year, with the title *An Essay on Canonical Forms.*

COMBINANTS. (Page. 161).

The theory of combinants is Sylvester's, *Cambridge and Dublin Mathematical Journal* (1853), vol. VIII., p. 63. In the case of the resultant of two equations it had, I think, been previously shown by Jacobi, that the resultant of $\lambda u + \mu v$, $\lambda' u + \mu' v$ was the resultant of u, v multiplied by a power of $(\lambda\mu' - \lambda'\mu)$. Sylvester's results Arts. 185, 188, 189, are given in the *Comptes rendus*, vol. LVIII., p. 1074–9.

APPLICATIONS TO BINARY QUANTICS. (Page 175).

In Lesson XVII the discussion of the quadratic, cubic, and quartic, is mainly Prof. Cayley's. See his Memoirs on Quantics in the *Philosophical Transactions*, 1854. The second form of the resultant of two quadratics, p. 180, is, as elsewhere stated, Dr. Boole's. Sylvester proved (*Philosophical Magazine*, April, 1853), that every invariant of a quartic is a rational function of S and T. The theorem, Art. 206, that the quartic may be reduced to its canonical form by real substitutions, is Legendre's (*Traité des Fonctions Elliptiques*, chap. II). The discussion of the systems of quadratic and cubic, two cubics, and two quartics, was, I believe, for the most part new, when it appeared in the second edition in 1866. The form for the resultant of two cubics, obtained by him by a different method, was published by Clebsch (*Crelle*, vol. LXIV.), but had been previously in my possession by the method given in Art. 213. On the connection p. 174 between concomitants of binary systems and those of a larger number of variables, R. Sturm's paper (Borchardt, LXXXVI. pp. 116–46) should be referred to. Also for the reduction of the system of two quartics, p. 224, announced by Sylvester, see Stroh, *Math. Ann.* XXII. 293, who cites d'Ovidio as having also effected it.

In Lesson XVIII the canonical form of the quintic $ax^5 + by^5 + cz^5$, which so much facilitates its discussion, was given by Sylvester in his Essay on Canonical Forms, 1851. The invariants J and K were calculated by Prof. Cayley. The value of the discriminant and its resolution into the sum of products (p. 230) was given by me in 1850 (*Cambridge and Dublin Mathematical Journal*, vol. V. p. 154). Some most important steps in the theory of the quintic were made in Hermite's paper in the *Cambridge and Dublin Mathematical Journal*, 1854, vol. IX., p. 172, where the number of independent invariants was established; the invariant I was discovered; attention was called to the linear covariants; and the possibility demonstrated of expressing by invariants the conditions of the reality of the roots of all equations of odd degrees. The theory of the quintic was further advanced by Sylvester's "Trilogy" (*Philosophical Transactions*, 1864, p. 579); and in Hermite's series of papers in the first volume of the *Comptes rendus* for 1866 already referred to. The values of the invariants A, B, C of the sextic were given by Prof. Cayley in his papers on Quantics,

and the existence of the invariant E pointed out. The rest of what is stated in the text about the sextic in Arts 252–6, 260–1 is nearly as it appeared for the first time in the second edition. Arts. 257–9 are from Clebsch "*Theorie*, &c.," p. 297, and Arts. 262 to the end of the Lesson are from the sources indicated in the foot-notes.

The term "apolar" is due to Th. Reye, whose investigations on "Moments of Inertia, &c.," Borchardt, *Journal* LXXII, led to his "Erweiterung der Polarentheorie algebraischer Flächen," vol. LXXVIII, p. 97, which he opens by remarking, that the polar theory of surfaces of the n^{th} order has hitherto dealt only with polars of points taken singly or in groups. But in regard to any such surface F_n there also corresponds to any surface of class k, for $k < n$, a definite surface of order $n - k$ which becomes identical with the polar of a group of k points in case the surface of class k reduces to such a group. He claims a special interest for such surfaces of class k for which every arbitrary surface of order $n - k$ may be regarded as polar in regard to the surface F_n, and calls them "apolar to the surface F_n," because to them no definite polar belongs. Rosanes had previously (Borchardt, LXXVI, p. 313) termed two binary forms of the same degree whose quadriuvariant vanishes "conjugate to each other." To explain the relation between the terms and connect them with the process of transvection, let x_1, x_2 ... be any variables, and u_1, u_2, ... contragredient to them, so that when both are linearly transformed, the former by a direct and the latter by its reciprocal substitution the value of $u_1x_1 + u_2x_2 + u_3x_3 +$ is unchanged, then any relation among the former may be expressed symbolically by the vanishing of an expression $(a_1x_1 + a_2x_2 + a_3x_3 +)^n = a_x{}^n$, and any relation among the latter by that of $(a_1u_1 + a_2u_2 + a_3u_3 +)^p = a_u{}^p$. We may refer to these as the locus $a_x{}^n$ and the envelope $a_u{}^p$. Substituting for u_1, u_2, .. $\frac{d}{dx_1}$, $\frac{d}{dx_2}$, ... in $a_u{}^p$, supposing $p < n$, and operating with the result on $a_x{}^n$ a new locus is found, $a_a{}^p a_x{}^{n-p} = a_a{}^p a_x{}^{n-p}$, which is called a polar of $a_u{}^p$ in respect of $a_x{}^n$, it being obvious that polars of higher degrees may be formed by repeating the operation, if need be, on additional factors of the symbol $a_x{}^n$. Again, supposing $p > n$, and substituting for x_1, x_2, ... $\frac{d}{du_1}$, $\frac{d}{du_2}$, in $a_x{}^n$, and operating on $a_u{}^p$, we find a new envelope $a_a{}^n a_u{}^{p-n} = a_a{}^n a_u{}^{p-n}$, which should be similarly considered a pole of $a_x{}^n$ in respect of $a_u{}^p$.

When $n = p$, the derived function is an invariant whose vanishing $a_a{}^n = a_a{}^n = 0$ is the condition for $a_x{}^n = 0$, and $a_u{}^n = 0$ to be "conjugate."

If the form $a_a{}^n a_u{}^{p-n}$ or $a_a{}^n a_u{}^{p-n}$ vanish for all values of the variables u, $a_x{}^n$ is "apolar" to $a_u{}^p$, and if $c_x{}^p$ be any other function such that $a_x{}^n b_x{}^{p-n} = c_x{}^p$, then $a_c{}^p = 0$, or $a_u{}^p$ is conjugate to any form which contains $a_x{}^n$ as a factor.

TABLE OF A FEW TRANSVECTANTS OF A QUANTIC OF ANY DEGREE.

Let the quantic of degree ν be written with binomial coefficients ν_1, ν_2, &c.,

$$a = a_0x^\nu + \nu_1a_1x^{\nu-1}y + \nu_2a_2x^{\nu-2}y^2 +,$$

and let a covariant b whose source b_0 is a function of a_0, a_1, ... supposed cleared of any common numerical factor, so that it is in its lowest terms, be of the degree ρ, then with binomial coefficients $b = b_0x^\rho + \rho_1b_1x^{\rho-1}y +$.

The source may be named without suffix, thus, for the Hessian $h = a_0a_2 - a_1{}^2$,

$g = a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3$, $i = a_0a_4 - 4a_1a_3 + 3a_2{}^2$, $j = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3{}^2 - a_4a_1{}^2 - a_2{}^3$,

$k = a_0{}^2a_5 - 5a_0a_1a_4 + 2a_0a_2a_3 - 6a_1a_2{}^2 + 8a_1{}^2a_3$, $f = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3{}^2$,

$l = a_0a_2a_6 - 3a_0a_2a_5 + 2a_0a_4{}^2 - a_0a_1{}^2 + 3a_1a_2a_5 - a_1a_3a_4 - 3a_1a_2{}^2 + 2a_2a_3{}^2$,

$o = a_0{}^2a_7 - 7a_0a_1a_6 + 9a_0a_2a_5 - 5a_0a_3a_4 + 20a_1a_3{}^2 - 30a_1a_2a_4 + 12a_1{}^2a_5$,

$p = a_0a_8 - 8a_1a_7 + 28a_2a_6 - 56a_3a_5 + 35a_4{}^2$,

$q = 2a_0a_6(a_0a_4 - 4a_1a_3 + 3a_2{}^2) - 3a_0{}^2a_5{}^2 + 18a_0a_1a_4a_5 - 12a_0a_2a_3a_5 - 27a_1{}^2a_4{}^2$
$- 18a_0a_2a_4{}^2 + 108a_1a_2a_3a_4 - 54a_2{}^3a_4 + 16a_0a_3{}^2a_4 - 64a_1a_3{}^3 + 36a_2{}^2a_3{}^2 = 12(i_0i_2 - i_1{}^2)$,

$e = (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)\ a_5 - a_0^2a_4^2 + 3a_0a_1a_3a_4 + 4a_0a_2^2a_4 - 5a_1^2a_2a_4 - 4a_0a_2a_3^2$
$- 2a_1^2a_3^2 + 8a_1a_2^2a_3 - 3a_2^4$;

also let $a_0b_1 - a_1b_0 = (ab)_1$, $a_0b_2 - 2a_1b_1 + a_2b_0 = (ab)_2$, &c., then the transvectant

$2\,(ah)_1 = g$; $\quad 2\,(2\nu-5)\,(ah)_2 = (\nu-3)\,ia$; $\quad 4\,(2\nu-5)\,(ah)_3 = (\nu-4)\,k$;

$4\,(2\nu-5)\,(2\nu-7)\,(ah)_4 = (\nu-4)\,(\nu-5)\,fa + 12\,(\nu-1)\,(2\nu-7)\,j$;

$8\,(2\nu-5)\,(2\nu-7)\,(ah)_5 = (\nu-5)\,(\nu-6)\,o$;

$8\,(2\nu-5)(2\nu-7)(2\nu-9)\,(ah)_6 = (\nu-5)(\nu-6)(\nu-7)\,pa + 10\,(\nu-1)\,(2\nu-9)\,(5\nu-22)\,l$; &c.

$2\,(ai)_1 = k$; $\quad 2\,(2\nu-9)\,(ai)_2 = (\nu-5)\,fa - 6\,(2\nu-9)\,j$; $\quad 4\,(2\nu-9)\,(ai)_3 = (\nu-6)\,o$;

$4\,(2\nu-9)\,(2\nu-11)\,(ai)_4 = (\nu-6)\,(\nu-7)\,pa - 4\,(\nu-12)\,(2\nu-11)\,l$; &c.;

$2\,(hi)_1 = a_0^2a_2a_5 - a_0a_1^2a_5 - a_0^2a_3a_4 - 2a_0a_1a_2a_4 + 3a_1^3a_4 + 4a_0a_1a_3^2 - a_0a_2^2a_3 - 6a_1^2a_2a_3 + 3a_1a_2^3$
$= 3\,(aj)_1$; $\quad 2\,(2\nu-5)\,(2\nu-9)\,(hi)_2 = (\nu-5)\,(2\nu-5)\,la - (2\nu-7)\,i^2 + (2\nu-5)\,(\nu-6)\,e$;

$3\,(3\nu-13)\,(aj)_2 = (\nu-5)\,la + (3\nu-13)\,e$; &c.

The additional function of a_0, a_1 &c. appears on p. 237 as (9), where also e is (8) with altered sign, or $-M$ of Art. 235. The other sources h, g, i, k, f, o, p are the series given in the end of Art. 192; j is equally well known, and l and q begin with the sextic, Arts. 252, 257.

The 23 forms of the COMPLETE SYSTEM OF THE QUINTIC a may now be described as follows:

From a determine its covariants h, i, j, e, form also the linear covariant $\alpha = (ij)_2$ by operating with i on j; the quadricovariant τ of the cubic j; and the Jacobian $\mathfrak{S}$ of the quadrics i, τ. We have then

As *invariants:* the invariant of i, of i and τ, of τ, and the eliminant of $\mathfrak{S}$, α.

As *linear* covariants: α, and its Jacobians with i, τ, and $\mathfrak{S}$.

The *quadric* covariants: $i, \tau, \mathfrak{S}$. The *cubics:* j and its Jacobians with i and τ.

The *quartics:* e and its Jacobian with i. The *quintics:* a and its Jacobians with i, (k) and with τ. The *sextics:* h and its Jacobian with i. The *septic:* the Jacobian of h with j; and the *nonic:* the Jacobian of a with h, (g).

The 26 forms of the COMPLETE SYSTEM OF THE SEXTIC a may also be thus described:

From a determine its covariants h, i, j, l, q; form also the quadric covariants m, n (Art. 259), which are $(li)_2$, $(mi)_2$. We have then

The *invariants:* f, $(aj)_6$, the invariant of l, that of m, and (lmn).

The *quadrics:* l, m, n and their Jacobians λ, μ, ν.

The *quartics:* i, q, and the Jacobians of i with l, and of q with l and with m.

The *sextics:* a, j, and the Jacobians of a with l and with m, and of j with l.

The *octavics:* h, k, and the Jacobian of h with l. The *decimic:* the Jacobian of h with i; and the *duodecimic:* the Jacobian of a with h, (g).

In this notation we have the following SUMS OF POWERS OF DIFFERENCES OF THE ROOTS by help of which the first few terms in the equation for the squares of the differences can be calculated. They were given by M. Roberts (*Quarterly Journal*, vol. IV. p. 173), in whose papers will be found several interesting relations among the covariants of binary quantics.

$a^2\Sigma\,(\alpha-\beta)^2 = -\nu^2\,(\nu-1)\,h$, $\quad a^4\Sigma\,(\alpha-\beta)^4 = \nu^2\,(\nu-1)\,\{\nu^2h^2 - \tfrac{1}{6}\,(\nu-2)\,(\nu-3)\,a^2i\}$,

$a^6\Sigma\,(\alpha-\beta)^6 = \nu^2\,(\nu-1)\,\{-\nu^4h^3 + \tfrac{1}{4}\nu^2\,(\nu-2)\,(\nu-5)\,a^2hi - \tfrac{1}{4}\nu\,(\nu-2)\,(7\nu-15)\,a^2j$
$- \tfrac{1}{120}\,(\nu-2)\,(\nu-3)\,(\nu-4)\,(\nu-5)\,a^5f\}$,

$$a^5\Sigma\,(a-\beta)^8 = \nu^2\,(\nu-1)\left\{\nu^6h^4 - \tfrac{1}{2}\nu^4\,(\nu-2)\,(\nu-7)\,a^2h^2i - 2\nu^3\,(\nu-2)\,(3\nu-7)\,a^3hj\right.$$
$$+ \tfrac{1}{7}\tfrac{1}{2}\nu^2\,(\nu-2)\,(\nu-3)\,(\nu^2+8\nu-21)\,a^4i^2 + \tfrac{1}{9}\tfrac{1}{0}\nu^2\,(\nu-2)\,(\nu-3)\,(\nu-4)\,(\nu-21)\,a^4hf$$
$$\left. + \tfrac{1}{3}\nu\,(\nu-2)(\nu-3)(\nu-4)(3\nu-7)\,a^5l - \frac{(\nu-2)(\nu-3)(\nu-4)(\nu-5)(\nu-6)(\nu-7)}{2.3.4.5.6.7}\,a^6p\right\}.$$

TABLE OF EXPANDED RESULTANTS, CHIEFLY OF EQUATIONS WRITTEN WITHOUT BINOMIAL COEFFICIENTS.

1. The resultant of the two quadratics $(A, B, C\,\mathcal{X}\,x, y)^2$, $(a, b, c\,\mathcal{X}\,x, y)^2$

is $$(Ac - Ca)^2 - (Ab - Ba)\,(Bc - Cb),$$

or $$a^2C^2 - abBC + ac\,(B^2 - 2AC) + b^2AC - bcAB + c^2A^2.$$

2. The resultant of the quadratic $(A, B, C\,\mathcal{X}\,x, y)^2$ and the cubic $(a, b, c, d\,\mathcal{X}\,x, y)^3$, is

$$a^2C^3 - abBC^2 + acC\,(B^2 - 2AC) - ad\,(B^3 - 3ABC)$$
$$+ b^2AC^2 - bcABC + bdA\,(B^2 - 2AC) + c^2A^2C - cdBA^2 + d^2A^3.$$

3. The resultant of quadratic and quartic is

$$a^2C^4 - abBC^3 + acC^2\,(B^2 - 2AC) - adC\,(B^3 - 3ABC)$$
$$+ ae\,(B^4 - 4B^2AC + 2A^2C^2) + b^2AC^3 - bcABC^2 + bdAC\,(B^2 - 2AC)$$
$$- beA\,(B^3 - 3ABC) + c^2A^2C^2 - cdA^2BC + ceA^2\,(B^2 - 2AC) + d^2A^3C - deA^3B + e^2A^4.$$

4. The resultant of quadratic and quintic is

$$a^2C^5 - abBC^4 + acC^3\,(B^2 - 2AC) - adC^2\,(B^3 - 3ABC)$$
$$+ aeC\,(B^4 - 4AB^2C + 2A^2C^2) - af\,(B^5 - 5B^3AC + 5A^2BC^2) + b^2AC^4 - bcABC^3$$
$$+ bdAC^2\,(B^2 - 2AC) - beAC\,(B^3 - 3ABC) + bfA\,(B^4 - 4AB^2C + 2A^2C^2)$$
$$+ c^2A^2C^3 - cdA^2BC^2 + ceA^2C\,(B^2 - 2AC) - cfA^2\,(B^3 - 3ABC)$$
$$+ d^2A^3C^2 - deA^3BC + dfA^3\,(B^2 - 2AC) + e^2A^4C - efBA^4 + f^2A^5.$$

5. Discriminant of cubic is

$$27A^2D^2 + 4AC^3 + 4DB^3 - B^2C^2 - 18ABCD.$$

6. Resultant of two cubics $(A, B, C.\ D\,\mathcal{X}\,x, y)^3$, $(a, b, c, d\,\mathcal{X}\,x, y)^3$.

The value expressed in terms of the determinants of the form $Ab - Ba$ is given in p. 77 (and for forms with binomial coefficients, p. 207). Expanded it is

$$a^3D^3 - a^2bCD^2 + a^2cD\,(C^2 - 2BD) - a^2d\,(C^3 - 3BCD + 3AD^2) + ab^2BD^2$$
$$- abcD\,(BC - 3AD) + abd\,(BC^2 - 2B^2D - ACD) + ac^2D\,(B^2 - 2AC)$$
$$+ acd\,(2AC^2 + ABD - B^2C) + ad^2\,(B^3 - 3ABC + 3A^2D) - b^3AD^2 + b^2cACD$$
$$- b^2dA\,(C^2 - 2BD) - bc^2ABD + bcdA\,(BC - 3AD) - bd^2A\,(B^2 - 2AC)$$
$$+ c^3A^2D - c^2dA^2C + cd^2A^2B - d^3A^3.$$

7. The discriminant of a quartic written with binomial coefficients, expanded is

$$a^3e^3 - 12a^2bde^2 - 18a^2c^2e^2 + 54a^2cd^2e - 27a^2d^4 + 54ab^2ce^2 - 6ab^2d^2e - 180abc^2de$$
$$+ 108abcd^3 + 81ac^4e - 54ac^3d^2 - 27b^4e^2 + 108b^3cde - 64b^3d^3 - 54b^2c^3e + 36b^2c^2d^2.$$

8. The discriminant of a quartic written without binomial coefficients is

$$4\,(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3)^2,$$

or expanding and dividing by 27,

$$256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e$$
$$- 80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2.$$

9. The resultant of cubic $(A, B, C, D)(x, y)^3$ and quartic $(a, b, c, d, e)(x, y)^4$ is

$$a^3D^4 - a^2bCD^3 + a^2cD^2\,(C^2 - 2BD) - a^2dD\,(C^3 - 3BCD + 3AD^2)$$
$$+ a^2e\,(C^4 - 4BC^2D + 2B^2D^2 + 4ACD^2) + ab^2BD^3 - abcD^2\,(BC - 3AD)$$
$$+ abdD\,(BC^2 - 2B^2D - ACD) - abe\,(BC^3 - 3B^2CD - AC^2D + 5ABD^2)$$
$$+ ac^2D^2\,(B^2 - 2AC) + acdD\,(2AC^2 + ABD - B^2C)$$
$$+ ace\,(B^2C^2 - 2AC^3 - 2DB^3 + 4ABCD - 3A^2D^2)$$
$$+ ad^2D\,(B^3 - 3ABC + 3A^2D) - ade\,(B^3C - 3ABC^2 - AB^2D + 5A^2CD)$$
$$+ ae^2\,(B^4 - 4AB^2C + 2A^2C^2 + 4A^2BD) - b^3AD^3 + bc^2ACD^2$$
$$- b^2dAD\,(C^2 - 2BD) + b^2eA\,(C^3 - 3BCD + 3AD^2) - bc^2ABD^2$$
$$+ bcdAD\,(BC - 3AD) + bceA\,(2B^2D + ACD - BC^2) - bd^2AD\,(B^2 - 2AC)$$
$$+ bdeA\,(B^2C - 2AC^2 - ABD) - bc^2A^2\,(B^3 - 3ABC + 3A^2D)$$
$$+ c^3A^2D^2 - c^2dA^2CD + c^2eA^2\,(C^2 - 2BD) + cd^2A^2BD - cdeA^2\,(BC - 3AD)$$
$$+ ce^2A^2\,(B^2 - 2AC) - d^3A^3D + d^2eA^3C - de^2A^3B + e^3A^4.$$

10. The resultant of two quartics $(A, B, C, D, E)(x, y)^4$, $(a, b, c, d, e)(x, y)^4$ is (cf. pp. 220, 277),

$$a^4E^4 - a^3bDE^3 + a^3cE^2\,(D^2 - 2CE) - a^3dE\,(D^3 - 3CDE + 3BE^2)$$
$$+ a^3e\,(D^4 - 4CD^2E + 2C^2E^2 + 4BDE^2 - 4AE^3) + a^2b^2CE^3$$
$$- a^2bcE^2\,(CD - 3BE) + a^2bdE\,(CD^2 - 2C^2E - BDE + 4AE^2)$$
$$- a^2be\,(CD^3 - 3C^2DE - BD^2E + 5BCE^2 + ADE^2) + a^2c^2E^2\,(C^2 - 2BD)$$
$$- a^2cdE\,(C^2D - 2BD^2 - BCE + 5ADE)$$
$$+ a^2ce\,(C^2D^2 - 2BD^3 - 2C^3E + 4BCDE + 2AD^2E - 3B^2E^2 + 2ACE^2)$$
$$+ a^2d^2E\,(C^3 - 3BCD + 3AD^2 + 3B^2E - 3ACE)$$
$$- a^2de\,(C^3D - 3BCD^2 + 3AD^3 - BC^2E + 5B^2DE - 2ACDE - 5ABE^2)$$
$$+ a^2e^2\,(C^4 - 4BC^2D + 2B^2D^2 + 4ACD^2 + 4B^2CE - 2AC^2E - 9ABDE + 4A^2E^2)$$
$$- ab^3BE^3 + ab^2cE^2\,(BD - 4AE) - ab^2dE\,(BD^2 - 2BCE - ADE)$$
$$+ ab^2e\,(BD^3 - 3BCDE - AD^2E + 3B^2E^2 + 2ACE^2) - abc^2E^2\,(BC - 3AD)$$
$$+ abcdE\,(BCD - 3AD^2 - 3B^2E + 4ACE)$$
$$- abce\,(BCD^2 - 3AD^3 - 2BC^2E - B^2DE + 8ACDE - 2ABE^2)$$
$$- abd^2E\,(BC^2 - 2B^2D - ACD + 5ABE)$$
$$+ abde\,(BC^2D - 2B^2D^2 - ACD^2 - B^2CE + 10ABDE - 8A^2E^2)$$
$$- abe^2\,(BC^3 - 3B^2CD - AC^2D + 5ABD^2 + 3B^3E - 2ABCE - 5A^2DE)$$
$$+ ac^3E^2\,(B^2 - 2AC) - ac^2dE\,(B^2D - 2ACD - ABE)$$
$$+ ac^2e\,(B^2D^2 - 2ACD^2 - 2B^2CE + 4AC^2E - 4A^2E^2)$$
$$+ acd^2E\,(B^2C - 2AC^2 - ABD + 4A^2E)$$

$$
\begin{aligned}
&- acde\,(B^2CD - 2AC^2D - ABD^2 - 3B^3E + 8ABCE - 2A^2DE)\\
&+ ace^2\,(B^2C^2 - 2AC^3 - 2B^3D + 4ABCD - 3A^2D^2 + 2AB^2E + 2A^2CE)\\
&- ad^2E\,(B^3 - 3ABC + 3A^2D) + ad^2e\,(B^2D - 3ABCD + 3A^2D^2 - AB^2E + 2A^2CE)\\
&- ade^2\,(B^3C - 3ABC^2 - AB^2D + 5A^2CD + A^2BE)\\
&+ ae^3\,(B^4 - 4AB^2C + 2A^2C^2 + 4A^2BD - 4A^3E) + b^4AE^3 - b^3cADE^2\\
&+ b^3dAE\,(D^2 - 2CE) - b^3eA\,(D^3 - 3CDE + 3BE^2) + b^2c^2ACE^2 - b^2cdAE\,(CD - 3BE)\\
&+ b^2ceA\,(CD^2 - 2C^2E - BDE + 4AE^2) + b^2d^2AE\,(C^2 - 2BD + 2AE)\\
&- b^2deA\,(C^2D - 2BD^2 - BCE + 5ADE) + b^2e^2A\,(C^3 - 3BCD + 3AD^2 + 3B^2E - 3ACE)\\
&- bc^3ABE^2 + bc^2dAE\,(BD - 4AE) - bc^2eA\,(BD^2 - 2BCE - ADE)\\
&- bcd^2AE\,(BC - 3AD) + bcdeA\,(BCD - 3AD^2 - 3B^2E + 4ACE)\\
&- bce^2A\,(BC^2 - 2B^2D - ACD + 5ABE) + bd^3AE\,(B^2 - 2AC)\\
&- bd^2eA\,(B^2D - 2ACD - ABE) + bde^2A\,(B^2C - 2AC^2 - ABD + 4A^2E)\\
&- be^3A\,(B^3 - 3ABC + 3A^2D) + c^4A^2E^2 - c^3dA^2DE + c^3eA^2\,(D^2 - 2CE)\\
&+ c^2d^2A^2CE - c^2deA^2\,(CD - 3BE) + c^2e^2A^2\,(C^2 - 2BD)\\
&- cd^3A^2BE + cd^2eA^2\,(BD - 4AE) - cde^2A^2\,(BC - 3AD) + ce^3A^2\,(B^2 - 2AC)\\
&+ d^4A^3E - d^3eA^3D + d^2e^2A^3C - de^3A^3B + e^4A^4.
\end{aligned}
$$

TABLES OF SYMMETRIC FUNCTIONS

as calculated by Meyer Hirsch and verified by Prof. Cayley. They have been since extended to the eleventh degree by M. Faà de Bruno (see his *Thèorie des Formes Binaires*), and in the *American Journal of Mathematics* to the twelfth degree by Mr. W. P. Durfee, vol. v. p. 46 and p. 348, and to the thirteenth by Capt. P. A. MacMahon, vol. VI. p. 289. The equation is supposed to be $x^n + bx^{n-1} + cx^{n-2} + \&c. = 0$.

I. $\Sigma a = -b.$

II. $\Sigma a^2 = b^2 - 2c;\ \Sigma a\beta = c.$

III. $\Sigma a^3 = -b^3 + 3bc - 3d;\ \Sigma a^2\beta = -bc + 3d;\ \Sigma a\beta\gamma = -d.$

IV. $\Sigma a^4 = b^4 - 4b^2c + 2c^2 + 4bd - 4e;\ \Sigma a^3\beta = b^2c - 2c^2 - bd + 4e.$

$\Sigma a^2\beta^2 = c^2 - 2bd + 2e;\ \Sigma a^2\beta\gamma = bd - 4e;\ \Sigma a\beta\gamma\delta = e.$

V. $\Sigma a^5 = -b^5 + 5b^3c - 5bc^2 - 5b^2d + 5cd + 5be - 5f.$

$\Sigma a^4\beta = -b^3c + 3bc^2 + b^2d - 5cd - be + 5f.$

$\Sigma a^3\beta^2 = -bc^2 + 2b^2d + cd - 5be + 5f.$

$\Sigma a^3\beta\gamma = -b^2d + 2cd + be - 5f.$

$\Sigma a^2\beta^2\gamma = -cd + 3be - 5f;\ \Sigma a^2\beta\gamma\delta = -be + 5f;\ \Sigma a\beta\gamma\delta\varepsilon = -f.$

VI. $\Sigma a^6 = b^6 - 6b^4c + 9b^2c^2 - 2c^3 + 6b^3d - 12bcd + 3d^2 - 6b^2e + 6ce + 6bf - 6g.$

$\Sigma a^5\beta = b^4c - 4b^2c^2 + 2c^3 - b^3d + 7bcd - 3d^2 + b^2e - 6ce - bf + 6g.$

$\Sigma a^4\beta^2 = b^2c^2 - 2c^3 - 2b^3d + 4bcd - 3d^2 + 2b^2e + 2ce - 6bf + 6g.$

$\Sigma a^3\beta^3 = c^3 - 3bcd + 3d^2 + 3b^2e - 3ce - 3bf + 3g.$

$\Sigma a^4\beta\gamma = b^3d - 3bcd + 3d^2 - b^2e + 2ce + bf - 6g.$

$\Sigma a^3\beta^2\gamma = bcd - 3d^2 - 3b^2e + 4ce + 7bf - 12g.$

$\Sigma a^2\beta^2\gamma^2 = d^2 - 2ce + 2bf - 2g;\ \Sigma a^3\beta\gamma\delta = b^2e - 2ce - bf + 6g.$

$\Sigma a^2\beta^2\gamma\delta = ce - 4bf + 9g;\ \Sigma a^2\beta\gamma\delta\varepsilon = bf - 6g;\ \Sigma a\beta\gamma\delta\varepsilon\zeta = g.$

VII. $\Sigma\alpha^7 = -b^7 + 7b^5c - 14b^3c^2 + 7bc^3 - 7b^4d + 21b^2cd - 7c^2d - 7bd^2 + 7b^3e - 14bce + 7de - 7b^2f + 7cf + 7bg - 7h.$

$\Sigma\alpha^6\beta = -b^5c + 5b^3c^2 - 5bc^3 + b^4d - 9b^2cd + 7c^2d + 4bd^2 - b^3e + 8bce - 7de + b^2f - 7cf - bg + 7h.$

$\Sigma\alpha^5\beta^2 = -b^3c^2 + 3bc^3 + 2b^4d - 6b^2cd - 3c^2d + 7bd^2 - 2b^3e + 4bce - 7de + 2b^2f + 3cf - 7bg + 7h.$

$\Sigma\alpha^4\beta^3 = -bc^3 + 3b^2cd + c^2d - 5bd^2 - 3b^3e + 2bce + 5de + 7b^2f - 7cf - 7bg + 7h.$

$\Sigma\alpha^5\beta\gamma = -b^4d + 4b^2cd - 2c^2d - 4bd^2 + b^3e - 3bce + 7de - b^2f + 2cf + bg - 7h.$

$\Sigma\alpha^4\beta^2\gamma = -b^2cd + 2c^2d + bd^2 + 3b^3e - 8bce + 2de - 3b^2f + 4cf + 8bg - 14h.$

$\Sigma\alpha^3\beta^3\gamma = -c^2d + 2bd^2 + bce - 5de - 4b^2f + 7cf + 4bg - 7h.$

$\Sigma\alpha^3\beta^2\gamma^2 = -bd^2 + 2bce + de - 2b^2f - 3ef + 7bg - 7h.$

$\Sigma\alpha^4\beta\gamma\delta = -b^3e + 3bce - 3de + b^2f - 2cf - bg + 7h.$

$\Sigma\alpha^3\beta^2\gamma\delta = -bce + 3de + 4b^2f - 6cf - 9bg + 21h.$

$\Sigma\alpha^2\beta^2\gamma^2\delta = -de + 3cf - 5bg + 7h;\ \Sigma\alpha^3\beta\gamma\delta\varepsilon = -b^2f + 2cf + bg - 7h.$

$\Sigma\alpha^2\beta^2\gamma\delta\varepsilon = -cf + 5bg - 14h;\ \Sigma\alpha^2\beta\gamma\delta\varepsilon\zeta = -bg + 7h;\ \Sigma\alpha\beta\gamma\delta\varepsilon\zeta\eta = -h.$

VIII. $\Sigma\alpha^8 = b^8 - 8b^6c + 20b^4c^2 - 16b^2c^3 + 2c^4 + 8b^5d - 32b^3cd + 24bc^2d + 12b^2d^2 - 8cd^2 - 8b^4e + 24b^2ce - 8c^2e - 16bde + 4e^2 + 8b^3f - 16bcf + 8df - 8b^2g + 8cg + 8bh - 8i.$

$\Sigma\alpha^7\beta = b^6c - 6b^4c^2 + 9b^2c^3 - 2c^4 - b^5d + 11b^3cd - 17bc^2d - 5b^2d^2 + 8cd^2 + b^4e - 10b^2ce + 8c^2e + 9bde - 4e^2 - b^3f + 9bcf - 8df + b^2g - 8cg - bh + 8i.$

$\Sigma\alpha^6\beta^2 = b^4c^2 - 4b^2c^3 + 2c^4 - 2b^5d + 8b^3cd - 9b^2d^2 + 2cd^2 + 2b^4e - 6b^2ce - 4c^2e + 16bde - 4e^2 - 2b^3f + 4bcf - 8df + 2b^2g + 4cg - 8bh + 8i.$

$\Sigma\alpha^5\beta^3 = b^2c^3 - 2c^4 - 3b^3cd + 6bc^2d + 3b^2d^2 - 7cd^2 + 3b^4e - 9b^2ce + 8c^2e + bde - 4e^2 - 3b^3f + bcf + 7df + 8b^2g - 8cg - 8bh + 8i.$

$\Sigma\alpha^4\beta^4 = c^4 - 4bc^2d + 2b^2d^2 + 4cd^2 + 4b^2ce - 4c^2e - 8bde + 6e^2 - 4b^3f + 8bcf - 4df + 4b^2g - 4cg - 4bh + 4i.$

$\Sigma\alpha^6\beta\gamma = b^5d - 5b^3cd + 5bc^2d + 5b^2d^2 - 5cd^2 - b^4e + 4b^2ce - 2c^2e - 9bde + 4e^2 + b^3f - 3bcf + 8df - b^2g + 2cg + bh - 8i.$

$\Sigma\alpha^5\beta^2\gamma = b^3cd - 3bc^2d - b^2d^2 + 5cd^2 - 3b^4e + 11b^2ce - 4c^2e - 10bde + 8e^2 + 3b^3f - 8bcf + df - 3b^2g + 4cg + 9bh - 16i.$

$\Sigma\alpha^4\beta^3\gamma = bc^2d - 2b^2d^2 - cd^2 - b^2ce + 10bde - 8e^2 + 4b^3f - 10bcf + df - 9b^2g + 16cg + 9bh - 16i.$

$\Sigma\alpha^4\beta^2\gamma^2 = b^2d^2 - 2cd^2 - 2b^2ce + 4c^2e - 4e^2 + 2b^3f - 4bcf + 8df - 2b^2g - 4cg + 8bh - 8i.$

$\Sigma\alpha^3\beta^3\gamma^2 = cd^2 - 2c^2e - bde + 4e^2 + 5bcf - 7df - 5b^2g + 2cg + 8bh - 8i.$

$$\Sigma\alpha^5\beta\gamma\delta = b^4e - 4b^3ce + 2c^2e + 4bde - 4e^2 - b^3f + 3bcf - 3df + b^2g - 2cg - bh + 8i.$$

$$\Sigma\alpha^4\beta^2\gamma\delta = b^2ce - 2c^2e - bde + 4e^2 - 4b^3f + 11bcf - 9df + 4b^2g - 6cg - 10bh + 24i.$$

$$\Sigma\alpha^3\beta^3\gamma\delta = c^2e - 2bde + 2e^2 - bcf + 3df + 5b^2g - 9cg - 5bh + 12i.$$

$$\Sigma\alpha^3\beta^2\gamma^2\delta = bde - 4e^2 - 3bcf + 6df + 5b^2g - 17bh + 24i.$$

$$\Sigma\alpha^2\beta^2\gamma^2\delta^2 = e^2 - 2df + 2cg - 2bh + 2i.$$

$$\Sigma\alpha^4\beta\gamma\delta\varepsilon = b^3f - 3bcf + 3df - b^2g + 2cg + bh - 8i.$$

$$\Sigma\alpha^3\beta^2\gamma\delta\varepsilon = bcf - 3df - 5b^2g + 8cg + 11bh - 32i.$$

$$\Sigma\alpha^2\beta^2\gamma^2\delta\varepsilon = df - 4cg + 9bh - 16i\,;\quad \Sigma\alpha^3\beta\gamma\delta\varepsilon\zeta = b^2g - 2cg - bh + 8i.$$

$$\Sigma\alpha^2\beta^2\gamma\delta\varepsilon\zeta = cg - 6bh + 20i\,;\quad \Sigma\alpha^2\beta\gamma\delta\varepsilon\zeta\eta = bh - 8i\,;\quad \Sigma\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta = i.$$

IX.

$$\Sigma\alpha^9 = -b^9 + 9b^7c - 27b^5c^2 + 30b^3c^3 - 9bc^4 - 9b^6d + 45b^4cd - 54b^2c^2d + 9c^3d - 18b^3d^2 + 27bcd^2 - 3d^3 + 9b^5e - 36b^3ce + 27bc^2e + 27b^2de - 18cde - 9be^2 - 9b^4f + 27b^2cf - 9c^2f - 18bdf + 9ef + 9b^3g - 18bcg + 9dg - 9b^2h + 9ch + 9bi - 9j.$$

$$\Sigma\alpha^8\beta = -b^7c + 7b^5c^2 - 14b^3c^3 + 7bc^4 + b^6d - 13b^4cd + 30b^2c^2d - 9c^3d + 6b^3d^2 - 19bcd^2 + 3d^3 - b^5e + 12b^3ce - 19bc^2e - 11b^2de + 18cde + 5be^2 + b^4f - 11b^2cf + 9c^2f + 10bdf - 9ef - b^3g + 10bcg - 9dg + b^2h - 9ch - bi + 9j.$$

$$\Sigma\alpha^7\beta^2 = -b^5c^2 + 5b^3c^3 - 5bc^4 + 2b^6d - 10b^4cd + 5b^2c^2d + 5c^3d + 11b^3d^2 - 13bcd^2 + 3d^3 - 2b^5e + 8b^3ce + bc^2e - 20b^2de + 4cde + 9be^2 + 2b^4f - 6b^2cf - 5c^2f + 18bdf - 9ef - 2b^3g + 4bcg - 9dg + 2b^2h + 5ch - 9bi + 9j.$$

$$\Sigma\alpha^6\beta^3 = -b^3c^3 + 3bc^4 + 3b^4cd - 9b^2c^2d - 3c^3d - 3b^3d^2 + 18bcd^2 - 6d^3 - 3b^5e + 12b^3ce - 9bc^2e - 9b^2de + 9be^2 + 3b^4f - 9b^2cf + 9c^2f - 9ef - 3b^3g + 9dg + 9b^2h - 9ch - 9bi + 9j.$$

$$\Sigma\alpha^5\beta^4 = -bc^4 + 4b^2c^2d + c^3d - 2b^3d^2 - 7bcd^2 + 3d^3 - 4b^3ce + 3bc^2e + 13b^2de - 2cde - 11be^2 + 4b^4f - 7b^2cf - c^2f - 2bdf + 11ef - 9b^3g + 18bcg - 9dg + 9b^2h - 9ch - 9bi + 9j.$$

$$\Sigma\alpha^7\beta\gamma = -b^6d + 6b^4cd - 9b^2c^2d + 2c^3d - 6b^3d^2 + 12bcd^2 - 3d^3 + b^5e - 5b^3ce + 5bc^2e + 11b^2de - 11cde - 5be^2 - b^4f + 4b^2cf - 2c^2f - 10bdf + 9ef + b^3g - 3bcg + 9dg - b^2h + 2ch + bi - 9j.$$

$$\Sigma\alpha^6\beta^2\gamma = -b^4cd + 4b^2c^2d - 2c^3d + b^3d^2 - 7bcd^2 + 3d^3 + 3b^5e - 14b^3ce + 12bc^2e + 13b^2de - 4cde - 14be^2 - 3b^4f + 11b^2cf - 4c^2f - 10bdf + 18ef + 3b^3g - 8bcg - 3b^2h + 4ch + 10bi - 18j.$$

$$\Sigma\alpha^5\beta^3\gamma = -b^2c^2d + 2c^3d + 2b^3d^2 - 4bcd^2 + 3d^3 + b^3ce - 2bc^2e - 5b^2de + 2cde + 6be^2 - 4b^4f + 15b^2cf - 8c^2f - 5bdf - 2ef + 4b^3g - 10bcg - 10b^2h + 18ch + 10bi - 18j.$$

$$- 6b^2e^2 + 10ce^2 - b^5f + 13b^3ef - 21bc^2f - 12b^2df + 20cdf$$
$$+ 11bef - 5f^2 + b^4g - 12b^2cg + 10c^2g + 11bdg - 10eg$$
$$- b^3h + 11bch - 10dh + b^2i - 10ci - bj + 10k.$$

$\Sigma a^3\beta^2$ $= b^6c^2 - 6b^4c^3 + 9b^2c^4 - 2c^5 - 2b^7d + 12b^5cd - 12b^3c^2d - 8bc^3d$
$$- 13b^4d^2 + 28b^2cd^2 + c^2d^2 - 10bd^3 + 2b^5e - 10b^4ce$$
$$+ 4b^2c^2e + 6c^3e + 24b^3de - 28bcde + 10d^2e - 11b^2e^2$$
$$+ 2ce^2 - 2b^5f + 8b^3cf + 2bc^2f - 22b^2df + 4cdf + 20bef$$
$$- 5f^2 + 2b^4g - 6b^2cg - 6c^2g + 20bdg - 10eg - 2b^3h$$
$$+ 4bch - 10dh + 2b^2i + 6ci - 10bj + 10k.$$

$\Sigma a^7\beta^3$ $= b^4c^3 - 4b^2c^4 + 2c^5 - 3b^5cd + 12b^3c^2d - 2bc^3d + 3b^4d^2 - 24b^2cd^2$
$$+ 6c^2d^2 + 11bd^3 + 3b^5e - 15b^4ce + 18b^2c^2e - 10c^3e + 12b^3de$$
$$+ 3bcde - 11d^2e - 15b^2e^2 + 10ce^2 - 3b^5f + 12b^3cf - 9bc^2f$$
$$- 9b^2df - cdf + 20bef - 5f^2 + 3b^4g - 9b^2cg + 10c^2g - bdg$$
$$- 10eg - 3b^3h - bch + 11dh + 10b^2i - 10ci - 10bj + 10k.$$

$\Sigma a^6\beta^4$ $= b^2c^4 - 2c^5 - 4b^3c^2d + 8bc^3d + 2b^4d^2 - 9c^2d^2 + 2bd^3 + 4b^4ce$
$$- 12b^2c^2e + 10c^3e - 8b^3de + 12bcde - 2d^2e + 9b^2e^2 - 14ce^2$$
$$- 4b^5f + 16b^3cf - 18bc^2f - 6b^2df + 20cdf - 4bef - 5f^2$$
$$+ 4b^4g - 6b^2cg - 2c^2g - 4bdg + 14eg - 10b^3h + 20bch$$
$$- 10dh + 10b^2i - 10ci - 10bj + 10k.$$

$\Sigma a^5\beta^5$. $= c^5 - 5bc^3d + 5b^2cd^2 + 5c^2d^2 - 5bd^3 + 5b^2c^2e - 5c^3e - 5b^3de$
$$- 5bcde + 5d^2e + 5b^2e^2 + 5ce^2 - 5b^3cf + 10bc^2f + 10b^2df$$
$$- 15cdf - 15bef + 10f^2 + 5b^4g - 15b^2cg + 5c^2g + 10bdg$$
$$- 5eg - 5b^3h + 10bch - 5dh + 5b^2i - 5ci - 5bj + 5k.$$

$\Sigma a^5\beta\gamma$ $= b^7d - 7b^5cd + 14b^3c^2d - 7bc^3d + 7b^4d^2 - 21b^2cd^2 + 7c^2d^2$
$$+ 7bd^3 - b^6e + 6b^4ce - 9b^2c^2e + 2c^3e - 13b^3de + 26bcde$$
$$- 10d^2e + 6b^2e^2 - 6ce^2 + b^5f - 5b^3cf + 5bc^2f + 12b^2df$$
$$- 12cdf - 11bef + 5f^2 - b^4g + 4b^2cg - 2c^2g - 11bdg + 10eg$$
$$+ b^3h - 3bch + 10dh - b^2i + 2ci + bj - 10k.$$

$\Sigma a^7\beta^2\gamma$ $= b^5cd - 5b^3c^2d + 5bc^3d - b^4d^2 + 9b^2cd^2 - 7c^2d^2 - 4bd^3 - 3b^5e$
$$+ 17b^4ce - 23b^2c^2e + 4c^3e - 16b^3de + 21bcde + d^2e + 17b^2e^2$$
$$- 12ce^2 + 3b^5f - 14b^3ef + 12bc^2f + 13b^2df - 3cdf - 31bef$$
$$+ 10f^2 - 3b^4g + 11b^2cg - 4c^2g - 10bdg + 20eg + 3b^3h$$
$$- 8bch - dh - 3b^2i + 4ci + 11bj - 20k.$$

$\Sigma a^6\beta^2\gamma$ $= b^3c^2d - 3bc^3d - 2b^4d^2 + 6b^2cd^2 + 3c^2d^2 - 7bd^3 - b^4ce + 3b^2c^2e$
$$+ 5b^3de - 15bcde + 13d^2e - 3b^2e^2 + 4ce^2 + 4b^5f - 19b^3cf + 18bc^2f$$
$$+ 15b^2df - 19cdf - 7bef + 10f^2 - 4b^4g + 15b^2cg - 8c^2g - 4bdg$$
$$- 4eg + 4b^3h - 10bch - dh - 11b^2i + 20ci + 11bj - 20k.$$

$\Sigma a^5\beta^4\gamma$ $= bc^3d - 3b^2cd^2 - c^2d^2 + 5bd^3 - b^2c^2e + 5b^3de - 8d^2e - 8b^2e^2$
$$+ 4ce^2 + b^3cf - bc^2f - 12b^2df + 10cdf + 23bef - 15f^2$$

$\Sigma a^4\beta^4\gamma = -c^3d + 3bcd^2 - 3d^3 + bc^2e - 5b^2de + 2cde + 5be^2 - b^2cf + c^2f$
$+ 6bdf - 11ef + 5b^3g - 14bcg + 9dg - 5b^2h + 9ch + 5bi - 9j.$

$\Sigma a^3\beta^2\gamma^2 = -b^3d^2 + 3bcd^2 - 3d^3 + 2b^3ce - 6bc^2e + 6cde + be^2 - 2b^4f + 6b^2cf$
$- 8bdf - ef + 2b^3g - 4bcg + 9dg - 2b^2h - 5ch + 9bi - 9j.$

$\Sigma a^4\beta^3\gamma^2 = -bcd^2 + 3d^3 + 2bc^2e + b^2de - 8cde + 2be^2 - 5b^2cf + 6c^2f$
$+ 2bdf - 2ef + 5b^3g - 4bcg - 11b^2h + 4ch + 18bi - 18j.$

$\Sigma a^3\beta^3\gamma^3 = -d^3 + 3cde - 3be^2 - 3c^2f + 3bdf + 3ef + 3bcg - 6dg$
$- 3b^2h + 3ch + 3bi - 3j.$

$\Sigma a^6\beta\gamma\delta = -b^5e + 5b^3ce - 5bc^2e - 5b^2de + 5cde + 5be^2 + b^4f - 4b^2cf + 2c^2f$
$+ 4bdf - 9ef - b^3g + 3bcg - 3dg + b^2h - 2ch - bi + 9j.$

$\Sigma a^5\beta^2\gamma\delta = -b^3ce + 3bc^2e + b^2de - 5cde - be^2 + 4b^4f - 15b^2cf + 6c^2f + 15bdf$
$- 7ef - 4b^3g + 11bcg - 9dg + 4b^2h - 6ch - 11bi + 27j.$

$\Sigma a^4\beta^3\gamma\delta = -bc^2e + 2b^2de + cde - 5be^2 + b^2cf - 5bdf + 13ef - 5b^3g$
$+ 13bcg - 9dg + 11b^2h - 20ch - 11bi + 27j.$

$\Sigma a^4\beta^2\gamma^2\delta = -b^2de + 2cde + be^2 + 3b^2cf - 6c^2f - 2bdf + 3ef - 5b^3g$
$+ 12bcg - 9dg + 5b^2h + ch - 19bi + 27j.$

$\Sigma a^3\beta^3\gamma^2\delta = -cde + 3be^2 + 3c^2f - 4bdf - 7ef - 7bcg + 18dg + 12b^2h$
$- 13ch - 19bi + 27j.$

$\Sigma a^2\beta^2\gamma^2\delta^2 = -be^2 + 2bdf + ef - 2bcg - 3dg + 2b^2h + 5ch - 9bi + 9j.$

$\Sigma a^5\beta\gamma\delta\varepsilon = -b^4f + 4b^2cf - 2c^2f - 4bdf + 4ef + b^3g - 3bcg + 3dg - b^2h$
$+ 2ch + bi - 9j.$

$\Sigma a^4\beta^2\gamma\delta\varepsilon = -b^2cf + 2c^2f + bdf - 4ef + 5b^2g - 14bcg + 12dg - 5b^2h$
$+ 8ch + 12bi - 36j.$

$\Sigma a^3\beta^3\gamma\delta\varepsilon = -c^2f + 2bdf - 2ef + bcg - 3dg - 6b^2h + 11ch + 6bi - 18j.$

$\Sigma a^3\beta^2\gamma^2\delta\varepsilon = -bdf + 4ef + 4bcg - 9dg - 9b^2h + 5ch + 30bi - 54j.$

$\Sigma a^2\beta^2\gamma^2\delta^2\varepsilon = -ef + 3dg - 5ch + 7bi - 9j.$

$\Sigma a^4\beta\gamma\delta\varepsilon\zeta = -b^3g + 3bcg - 3dg + b^2h - 2ch - bi + 9j.$

$\Sigma a^3\beta^2\gamma\delta\varepsilon\zeta = -bcg + 3dg + 6b^2h - 10ch - 13bi + 45j.$

$\Sigma a^2\beta^2\gamma^2\delta\varepsilon\zeta = -dg + 5ch - 14bi + 30j;\ \Sigma a^3\beta\gamma\delta\varepsilon\zeta\eta = -b^2h + 2ch + bi - 9j.$

$\Sigma a^2\beta^2\gamma\delta\varepsilon\zeta\eta = -ch + 7bi - 27j;\ \Sigma a^2\beta\gamma\delta\varepsilon\zeta\eta\theta = -bi + 9j.$

X. $\Sigma a^{10} = b^{10} - 10b^8c + 35b^6c^2 - 50b^4c^3 + 25b^2c^4 - 2c^5 + 10b^7d - 60b^5cd$
$+ 100b^3c^2d - 40bc^3d + 25b^4d^2 - 60b^2cd^2 + 15c^2d^2 + 10bd^3$
$- 10b^6e + 50b^4ce - 60b^2c^2e + 10c^3e - 40b^3de + 60bcde - 10d^2e$
$+ 15b^2e^2 - 10ce^2 + 10b^5f - 40b^3cf + 30bc^2f + 30b^2df - 20cdf$
$- 20bef + 5f^2 - 10b^4g + 30b^2cg - 10c^2g - 20bdg + 10eg + 10b^3h$
$- 20bch + 10dh - 10b^2i + 10ci + 10bj - 10k.$

$\Sigma a^9\beta = b^9c - 8b^6c^2 + 20b^4c^3 - 16b^2c^4 + 2c^5 - b^7d + 15b^5cd - 46b^3c^2d$
$+ 31bc^3d - 7b^4d^2 + 33b^2cd^2 - 15c^2d^2 - 7bd^3 + b^6e$
$- 14b^4ce + 33b^2c^2e - 10c^3e + 13b^3de - 42bcde + 10d^2e$

$$- 5b^4g + 18b^2cg - 8c^2g - 7bdg - 4eg + 11b^3h - 31bch$$
$$+ 20dh - 11b^2i + 20ci + 11bj - 20k.$$

$$\Sigma\alpha^6\beta^2\gamma^2 = b^4d^2 - 4b^2cd^2 + 2c^2d^2 + 4bd^3 - 2b^4ce + 8b^2c^2e - 4c^3e - 8bcde$$
$$- 4d^2e - b^2e^2 + 10ce^2 + 2b^5f - 8b^3cf + 4bc^2f + 10b^2df$$
$$- 4cdf - 8bef + 5f^2 - 2b^4g + 6b^2cg - 8bdg - 2eg + 2b^3h$$
$$- 4bch + 10dh - 2b^2i - 6ci + 10bj - 10k.$$

$$\Sigma\alpha^5\beta^3\gamma^2 = b^2cd^2 - 2c^2d^2 - bd^3 - 2b^2c^2e + 4c^3e - b^3de + 5bcde + d^2e + b^2e^2$$
$$- 12ce^2 + 5b^3cf - 13bc^2f - 4b^2df + 17cdf + 10bef - 15f^2$$
$$- 5b^4g + 15b^2cg - 4c^2g - 19bdg + 20eg + 5b^3h - 3bch$$
$$- dh - 12b^2i + 4ci + 20bj - 20k.$$

$$\Sigma\alpha^4\beta^4\gamma^2 = c^2d^2 - 2bd^3 - 2c^3e + 4bcde + 2d^2e - 3b^2e^2 + 2ce^2 + 2bc^2f$$
$$+ 2b^2df - 12cdf + 4bef + 5f^2 - 6b^2cg + 10c^2g + 4bdg$$
$$- 14eg + 6b^3h - 12bch + 10dh - 6b^2i + 2ci + 10bj - 10k.$$

$$\Sigma\alpha^4\beta^3\gamma^3 = bd^3 - 3bcde - d^2e + 3b^2e^2 + 2ce^2 + 3bc^2f - 3b^2df + cdf - 8bef$$
$$+ 5f^2 - 3b^2cg - 4c^2g + 13bdg - 2eg + 3b^3h + bch - 11dh$$
$$- 10b^2i + 10ci + 10bj - 10k.$$

$$\Sigma\alpha^7\beta\gamma\delta = b^6e - 6b^4ce + 9b^2c^2e - 2c^3e + 6b^3de - 12bcde + 3d^2e - 6b^2e^2$$
$$+ 6ce^2 - b^5f + 5b^3cf - 5bc^2f - 5b^2df + 5cdf + 11bef - 5f^2$$
$$+ b^4g - 4b^2cg + 2c^2g + 4bdg - 10eg - b^3h + 3bch - 3dh$$
$$+ b^2i - 2ci - bj + 10k.$$

$$\Sigma\alpha^6\beta^2\gamma\delta = b^4ce - 4b^2c^2e + 2c^3e - b^3de + 7bcde - 3d^2e + b^2e^2 - 6ce^2 - 4b^5f$$
$$+ 19b^3cf - 17bc^2f - 19b^2df + 15cdf + 18bef - 15f^2 + 4b^4g$$
$$- 15b^2cg + 6c^2g + 15bdg - 6eg - 4b^3h + 11bch - 9dh$$
$$+ 4b^2i - 6ci - 12bj + 30k.$$

$$\Sigma\alpha^5\beta^3\gamma\delta = b^2c^2e - 2c^3e - 2b^3de + 4bcde - 3d^2e + 2b^2e^2 + 2ce^2 - b^3cf + 2bc^2f$$
$$+ 3b^2df - 4cdf - 12bef + 10f^2 + 5b^4g - 19b^2cg + 10c^2g + 15bdg$$
$$- 6eg - 5b^3h + 13bch - 9dh + 12b^2i - 22ci - 12bj + 30k.$$

$$\Sigma\alpha^4\beta^4\gamma\delta = c^3e - 3bcde + 3d^2e + 3b^2e^2 - 3ce^2 - bc^2f + 2b^2df + cdf - 8bef$$
$$+ 5f^2 + b^2cg - c^2g - 3bdg + 9eg - 6b^3h + 17bch - 15dh$$
$$+ 6b^2i - 11ci - 6bj + 15k.$$

$$\Sigma\alpha^5\beta^2\gamma^2\delta = b^3de - 3bcde + 3d^2e - b^2e^2 + 2ce^2 - 3b^3cf + 9bc^2f + 2b^2df$$
$$- 13cdf - bef + 10f^2 + 5b^4g - 17b^2cg + 4c^2g + 18bdg$$
$$- 18eg - 5b^3h + 12bch - 9dh + 5b^2i + 2ci - 21bj + 30k.$$

$$\Sigma\alpha^4\beta^3\gamma^2\delta = bcde - 3d^2e - 3b^2e^2 + 4ce^2 - 3bc^2f + 4b^2df + 5cdf - 5f^2$$
$$+ 7b^2cg - 8c^2g - 15bdg + 12eg - 12b^3h + 21bch + 3dh$$
$$+ 26b^2i - 28ci - 42bj + 60k.$$

$$\Sigma\alpha^3\beta^3\gamma^3\delta = d^2e - 2ce^2 - cdf + 5bef - 5f^2 + 4c^2g - 7bdg + 2eg - 4bch$$
$$+ 11dh + 7b^2i - 10ci - 7bj + 10k.$$

$$\Sigma\alpha^4\beta^2\gamma^2\delta^2 = b^2e^2 - 2ce^2 - 2b^2df + 4cdf - 5f^2 + 2b^2cg - 4c^2g + 10eg - 2b^3h + 4bch - 10dh + 2b^2i + 6ci - 10bj + 10k.$$

$$\Sigma\alpha^3\beta^3\gamma^2\delta^2 = ce^2 - 2cdf - bef + 5f^2 + 2c^2g + 3bdg - 9eg - 7bch + 6dh + 7b^2i + ci - 15bj + 15k.$$

$$\Sigma\alpha^6\beta\gamma\delta\varepsilon = b^3f - 5b^2cf + 5bc^2f + 5b^2df - 5cdf - 5bef + 5f^2 - b^4g + 4b^2cg - 2c^2g - 4bdg + 4eg + b^3h - 3bch + 3dh - b^2i + 2ei + bj - 10k.$$

$$\Sigma\alpha^5\beta^2\gamma\delta\varepsilon = b^2ef - 3bc^2f - b^2df + 5cdf + bef - 5f^2 - 5b^4g + 19b^2cg - 8c^2g - 19bdg + 16eg + 5b^3h - 14bch + 12dh - 5b^2i + 8ci + 13bj - 40k.$$

$$\Sigma\alpha^4\beta^3\gamma\delta\varepsilon = bc^2f - 2b^2df - cdf + 5bef - 5f^2 - b^2cg + 5bdg - 8eg + 6b^3h - 16bch + 12dh - 13b^2i + 24ci + 13bj - 40k.$$

$$\Sigma\alpha^4\beta^2\gamma^2\delta\varepsilon = b^2df - 2cdf - bef + 5f^2 - 4b^2cg + 8c^2g + 3bdg - 12eg + 9b^3h - 23bch + 18dh - 9b^2i + 4ci + 33bj - 60k.$$

$$\Sigma\alpha^3\beta^3\gamma^2\delta\varepsilon = cdf - 3bef + 5f^2 - 4c^2g + 6bdg + 9bch - 24dh - 21b^2i + 28ci + 33bj - 60k.$$

$$\Sigma\alpha^3\beta^2\gamma^2\delta^2\varepsilon = bef - 5f^2 - 3bdg + 8eg + 5bch - 2dh - 7b^2i - 8ci + 31bj - 40k.$$

$$\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2 = f^2 - 2eg + 2dh - 2ci + 2bj - 2k.$$

$$\Sigma\alpha^5\beta\gamma\delta\varepsilon\zeta = b^4g - 4b^2cg + 2c^2g + 4bdg - 4eg - b^3h + 3bch - 3dh + b^2i - 2ci - bj + 10k.$$

$$\Sigma\alpha^4\beta^2\gamma\delta\varepsilon\zeta = b^2cg - 2c^2g - bdg + 4eg - 6b^3h + 17bch - 15dh + 6b^2i - 10ci - 14bj + 50k.$$

$$\Sigma\alpha^3\beta^3\gamma\delta\varepsilon\zeta = c^2g - 2bdg + 2eg - bch + 3dh + 7b^2i - 13ci - 7bj + 25k.$$

$$\Sigma\alpha^3\beta^2\gamma^2\delta\varepsilon\zeta = bdg - 4eg - 5bch + 12dh + 14b^2i - 12ci - 46bj + 100k.$$

$$\Sigma\alpha^2\beta^2\gamma^2\delta^2\varepsilon\zeta = eg - 4dh + 9ci - 16bj + 25k.$$

$$\Sigma\alpha^4\beta\gamma\delta\varepsilon\zeta\eta = b^3h - 3bch + 3dh - b^2i + 2ci + bj - 10k.$$

$$\Sigma\alpha^3\beta^2\gamma\delta\varepsilon\zeta\eta = bch - 3dh - 7b^2i + 12ci + 15bj - 60k.$$

$$\Sigma\alpha^2\beta^2\gamma^2\delta\varepsilon\zeta\eta = dh - 6ci + 20bj - 50k\,;\quad \Sigma\alpha^3\beta\gamma\delta\varepsilon\zeta\eta\theta = b^2i - 2ci - bj + 10k.$$

$$\Sigma\alpha^2\beta^2\gamma\delta \text{ \&c.} = ci - 8bj + 35k\,;\quad \Sigma\alpha^2\beta\gamma \text{ \&c.} = bj - 10k.$$

Prof. Cayley noticed a certain symmetry in the coefficients of the preceding formulæ, which may be more easily exhibited by using Hirsch's notation. If such a sum as $\Sigma\alpha^3\beta^2\gamma^2\delta\varepsilon\zeta$ be denoted $[32^21^3]$ and the coefficients be a_1, a_2, &c., so that (32^21^3) will denote $a_3a_2^2a_1^3$, then the formulæ for the sums of the fourth order may be written

	(4)	(31)	(2²)	(21²)	(1⁴)
[4] =	−4	+4	+2	−4	+1
[31] =	+4	−1	−2	+1	
[2²] =	+2	−2	+1		
[21²] =	−4	+1			
[1⁴] =	+1				

The first line is to be read $\Sigma\alpha^4 = -4a_4 + 4a_3a_1 + 2a_2^2 - 4a_2a_1^2 + a_1^4$, and so on for the rest. Now what Prof. Cayley has proved is, that when the formulæ already given are thus written, the figures are the same whether we read according to the rows or columns. The same thing holds for Prof. Cayley's (*Phil. Trans.*, 1857, p. 489) formulæ expressing the coefficients (4), (31), &c. in terms of the sums [4], [31], &c.

INDEX.

Absolute invariants, 111, 175.
Apolarity, 224, 336, 337, 346.
Aronhold, on symbolical methods, 147.
On invariants of ternary cubic, 343.
On the differential equations of invariants, 344.

Baltzer, on determinants, 338.
Bezout, on elimination, 81, 105, 338, 342.
Bezoutiants, 343.
Binet, on determinants, 338.
Boole, on linear transformations, 109, 343.
Form for the resultant of two quadratics, 24, 165, 180.
Borchardt, proof that the equation of the secular inequalities has all its roots real, 54.
Bordered Hessians reduced, 17.
Bordered symmetrical determinants, value of, 36.
Skew symmetrical determinants, 266.
Brill, on sextic, 273.
Brioschi expresses differential equation of invariants in terms of roots, 342.
On solution of the quintic, 257.
On determinants, 339.
Burnside, investigation of radius of sphere circumscribing tetrahedron, 26.
Transformation of binary to ternary forms, 172.
Applications of this method, 181, 189, 200, 202, 214, 222, 281.
On solution of biquadratic, 196.

Canonical forms, 150, 194, 228, 277, 281, 345.
Canonizants, 154, 226, 282.
Catalecticants, 156, 190, 261.
Cauchy, on determinants, 33, 338, 340.
Cayley (see also p. 343).
His expression for relation connecting mutual distances of five points on a sphere, 26.
of five points in space, 27.
Application of skew determinants to the theory of orthogonal substitutions, 41, 339.
Calculation of number of terms in a symmetrical determinant, 45.
On symmetric functions of roots of equation, 58, 342.
Statement of Bezout's method of elimination, 83.
General expression for resultants as quotients of determinants, 87, 306.
Notation for quantics, 99.
Cayley, discovery of invariants, 109, 343.
On the number of invariants of a binary quantic, 132, 176.
Definition of covariants, 136.
Symbolical method of expressing invariants and covariants, 137.
Identifies two forms of canonizant of equations of odd degree, 155.
On discriminant of discriminant, 166.
On tact-invariants, 169.
Method of forming a complete system of covariants, 176, 186, 199.
Relation connecting covariants of cubic, 186.
Solution of a cubic, 186.
Solution of a quartic, 195.
On criteria of reality of roots, 197.
On covariants of system formed by quartic and its Hessian, 201.
On covariants of quintic, 237.
Canonical form for quintic, 238.
Tables of Sturmian functions, 240.
On Tschirnhausen transformation, 251.
On rational functional determinants, 340.
On invariants of sextic, 345.
Tables of symmetric functions, 350, 356.
Clebsch, on symbolical methods, 147.
Special form of ternary quartics, 151, 335.
Conditions a sextic may be sexti-covariant of quartic, 282.
Proves that every invariant may be symbolically expressed, 315.
Proves number of forms is finite, 320.
Investigates resultant of quadratic and general equation, 326.
General expression for discriminant, 326.
Investigates equation of system of inflexional tangents to a cubic, 330.
On type form of even binary, 335,
Form of resultant of two cubics, 345.
Cockle, on the solution of the quintic, 257.
Combinants, 161, 345.
Invariant of invariant of $u + \lambda v$ is a combinant, 211.
Of a system of two quartics, 219.
Common roots determined, 91.
Commutants, 339.
Complete systems, 177, 185, 199, 237, 260, 347.
Concomitants, 121.
Conditions that equations should have two common factors, 78, 97, 291, 312.

Conditions for systems of equalities between roots, 126.
That quantic should be reducible to sum of powers, 156.
That $u + \lambda v$ should have a cubic factor, 166, 205, 221, 279.
That $u + \lambda v$ have two double factors, 166, 220, 279.
That four points should form a harmonic system, 179.
That three pairs of points should form system in involution, 180.
For three quadrics to be differentials of a quartic, 224.
to be quadric covariants of two cubics, 225.
That quartic should have two square factors, 220.
That two quartics should be differentials of same quintic, 220, 228.
That quintic should admit of being brought by linear transformation to Jerrard's form, 232.
That quintic should have two square or a cubic factor, 233.
That sextic should have two square or a cubic factor, 263.
That roots of sextic should be in involution, 270.
For sextic to be Hessian of a quintic, 282.
For sextic to be sexticovariant of quartic, 282.
That quantic have two square or one cubic factor, 294.
Conjugate forms, 225, 337, 346.
Contragredience, 118, 346.
Contravariants, 117, 120.
Of binary quantics not essentially distinct from covariants, 127.
Continuants, 18.
Covariants, 114; distinct, 175.
How defined by Cayley, 136.
Number of, for binary quantic, 132, 176.
Cramer, on determinants, 338.
Critical functions, 60.
Cubicovariant of cubic, 130, 183.
Cubic discussed, 183.
System of two, 204; of four, 215, 337.
Cubic, quaternary, its canonical form, 160.
Cubinvariants, only type of, 141.

Derivatives of derivatives expressed symbolically, 321.
Dialytic method of elimination, 79, 342.
Differential coefficients of determinants, 35.
Of resultants with respect to quantities entering into all the quantics, 96.
Differential equation of functions of differences of roots, 61, 342.
Of invariants, 131.
Differentiation mutual, of covariants and contravariants, 126.
Discriminant defined, 99.
Of binary quantic expressed as determinant, 23.
Discriminant of product of two quantics, 101, 167.
Of discriminants, 166.
Sign of distinguishes whether equation has even or odd numbers of pairs of imaginary roots, 239.
General symbolical expression for, 326.
Distinct invariants and covariants, 175.
Double points of involution, 162.
Double tangents of plane curves, 333.
D'Ovidio, on two cubics, 213; quartics, 345.
Durfee, on symmetric functions, 350.

Eisenstein, expression for general solution of quartic, 343.
Eliminants defined, 66.
Elimination, 67.
Emanants, 115, 333.
Equalities between roots of an equation, conditions for, 125.
Equianharmonic, 184.
Invariant, 190, 259.
Euler, a theorem of, 25.
On the theory of orthogonal substitutions, 44.
On elimination, 77, 342.
Evectants, 122.
Of discriminant which vanishes, 123.
Symbolical expression for, 146.
Of discriminant of cubic, 130, 183.
Of cubinvariant of quartic, 192.
Of quartinvariant of quintic, 237.
Of sextic, 279.

Faà de Bruno, calculates invariant of quintic, 230.
On elimination, 342.
On symmetric functions, 350.
Forme-type of quintic, 249.
Of even quantic, 335.
Of quantic of order $3p$, 337.

Gauss, on linear transformations, 338, 343.
Gerbaldi, on two cubics, 213.
Gordan, on number of covariants, 176, 213, 224, 260, 320, 324.
Gundelfinger, on system of cubic and quartic, 218.

Harley, on solution of a quintic, 257.
Harmonic invariant, 190, 260.
Hermite, law of reciprocity, 142, 179.
On transformation of a quadratic function, 42.
On concomitants of system formed by quartic and its Hessian, 201.
Canonical form for quintic, 233.
Discovery of skew invariant of quintic 233, 345.
Expression by invariants of conditions of reality of roots, 242, 250.
Forme-type of quintic, 249.
On Tschirnhausen transformation, 251.
Solution of quintics by elliptic functions, 257.
Expression of invariants of quintic in terms of roots, 258.

Hessians, 17, 117, 144, 183, 192, 227, 261, 341.
Contain all square factors of original quantic, 153, 198.
Of Hessians, 202, 321.
Hill, on quintic, 260.
Hirsch's tables of symmetric functions, 350.
Homographic determinant, 33.
Hyper-determinants, 343.

Inflexional tangents to cubic, calculation of their equation, 330.
Invariants, 111; irreducible, 175.
Absolute, 111, 190.
Skew, 131.
How many independent, 111.
Relation connecting weight and order of, 130.
Involution, 162.
Condition roots of sextic be in, 270.
Determinant, 25, 33, 180, 270.
Foci of, 179, 193, 259.

Jacobi, on determinants and linear transformations, 339, 340, 341.
Jacobian, of systems of equations, 84, 117, 342.
Properties of, 84.
Geometrically interpreted, 162.
Its discriminant discussed, 164.
Of two quadratics, 179.
Of quartic and its Hessian, 201.
Of systems of quartics, 271, 274.
Jerrard, transformation of a quintic, 232, 250, 259.
Joachimsthal, expression for area of a triangle inscribed in an ellipse, 26.
Theorem on form of discriminant, 102.

Kronecker, solution of quintic by elliptic functions, 257.
Kummer's resolution into sum of squares of discriminant of cubic which determines axes of a quadric, 55.

Lagrange, on solution of quintic, 257.
On determinants, 338.
On conditions that equation should have two pairs of equal roots, 342.
On linear transformations, 343.
Laplace, on determinants, 338.
Equation of secular inequalities, 48.
Leibnitz, his claim to invention of determinants, 338.
Lindemann, on geometric exposition, 174, 216, 218, 336.
Condition for sextic to be Hessian of a quintic, 282.
Linear covariants of cubic and quadratic, 187.
Of two cubics, 209, 215.
Of cubic and quartic, 219.
Of quintic, 235, 249.

MacMahon, on symmetric functions, 350.
Meyer, on apolarity, 224, 276.
Minor determinants, 10, 28.
Minor determinants, of reciprocal system how related to those of original, 31.
Muir, on continuants, 18.
Multiplication of determinants, 20.

Newton, on sums of powers of roots of equation, 56.
Number of terms in a symmetrical determinant, 45.
Of quadrics which can be described through five points to touch four planes, 298.
Of invariants of a binary quantic, 132, 175.
Of distinct forms finite, 176, 325.

Order of determinants, 7.
Of symmetric functions, 58.
Of invariants, 130.
Of resultant of any equations, 76.
Of discriminants, 99.
Of systems of equations, 284.
Orthogonal substitutions, 42.
Osculants, 171.

Poisson's method of forming symmetric functions of common roots of systems of equations, 72, 342.

Quadratic forms, transformed, 42.
Reducible to sum of squares, 151.
Number of positive and negative squares fixed, 151.
General expression for resultant with equation of n^{th} degree, 327.
Quadric systems, 179.
Quadrinvariants of binary quantics, 128.
Quartic, theory of, 189.
System of two, 219, 271.
Quartinvariant of odd quantic, 129.
Quintic, theory of, 227, 347.
Involved in theory of sextic, 273.

Reciprocal determinants, 30.
Reciprocity, Hermite's law of, 142, 179.
Reducing sextic for quintic, forms of, 257.
Resultant, order and weight of, 66, 75.
Of two quadratics, 63, 77, 83, 85, 180.
Of two cubics, 77, 85, 205, 345.
Of two quartics, 81, 86, 220, 277.
Of quadratic and any equation, 327.
Tables of, 348.
Reye, on apolarity, 346.
Roberts, Michael, on sources of covariants, 134, 238.
On application of Sturm's theorem to quintics, 240.
On equation of squares of differences, 347.
Roberts, Samuel, on orders of systems of equations, 310.
Roberts, W. R. W., on twisted cubic, 174, 219*c*, *d*.
Rodrigues, orthogonal transformations, 44.
Rosanes, on conjugate forms, 346.

Schlesinger, on conjugate forms, 225.
Seminvariants, 176.
Serret's notation for differential equation of covariants, 65.
Sextic, theory of, 260, 347.
Skew symmetric determinants, of even degree are perfect squares, 38.
Bordered, 266.
Skew invariants defined, 131.
Of all quantics vanish when quantic wants alternate terms, 234.
Skew invariant of quintic, 233, 282.
Vanishes if quintic can be linearly transformed to recurring form, 242.
Expression in terms of roots, 258.
Skew invariant of sextic, 260, 269, 282.
Source of covariants, 134, 238.
Sphere circumscribing tetrahedron, 26.
Relations connecting mutual distances of points on, 26.
Spottiswoode, on determinants, 338.
Stephanos, on sextic, 273, 282.
Stroh, on two quartics, 345.
Sturm's functions, Sylvester's expressions for, 49.
In case of quartic, 197.
of quintic, 240.
Extension of, 105.
Sturm, R., on geometrical representation, 345.
Superfluous variable, method of using, 207.
Sylvester (see also p. 343).
Umbral notation for determinants, 8.
Proof that equation of secular inequalities has all real roots, 28, 48.
Expression for Sturm's functions in terms of roots, 49.
Dialytic method of elimination, 79, 342.
Expression of resultant as determinant, 86.
Sylvester, extension of Sturm's theorem, 105.
On nomenclature, 121, 122, 190, 341, 342, 345.
Canonical forms of odd and even degrees, 153, 156, 227, 345.
Of quaternary cubic, 160.
Expressions for discriminant with regard to variables which do not enter explicitly, 168.
On osculants, 171.
On number of distinct forms, 176.
Reduces system of two cubics, 213.
Reduces system of cubic and quartic, 218.
Reduces system of two quartics, 224, 345.
Investigation of invariant conditions for reality of roots of quintic, 242.
On Bezoutiants, 343.
On combinants, 345.
Symbolical expression for invariants, &c., 137, 314.
Symmetric functions, 56.
Their use in finding invariants, 124.
Tables of, 350.

Tact-invariants, 169.
Of complex curves, 170.
Tetrahedron, radius of circumscribing sphere, 26.
Transvection and transvectants, 320, 346.
Tschirnhausen, transformation of equations, 250.
Type, or typical forms, 249, 335.

Umbral notation, 8, 314, 338.

Vandermonde, on determinants, 338.

Warren, on system of two quartics, 221.

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